1. In a binary tree all nodes are either internal or they are leaves. In our definition, internal nodes always have two children and leaves have zero children. Prove that for such trees, the number of leaves is always one more than the number of internal nodes.

There are many possible proofs based on induction, but there might be others.

2. For \( n \geq 0 \) consider \( 2^n \times 2^n \) matrices of 1s and 0s in which all elements are 1, except one which is 0 (The 0 is at an arbitrary position).

Operation: At each step, we can replace three 1s forming an "L" with three 0s (The L’s can have an arbitrary orientation).

Example of an \( 4 \times 4 \) matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Prove that for such matrices there always is a sequence of operations that transforms the matrix to the all 0 matrix.

3. Suppose you are given an array \( A \) of \( n \) distinct integers with the following property: There exists a unique index \( p \) such that the values of \( A[1 \ldots p] \) are decreasing and the values of \( A[p \ldots n] \) are is increasing.

For instance, in the example below we have \( n = 10 \) and \( p = 4 \):

\[ A = [20, 18, 12, 10, 50, 51, 69, 94, 103, 111] \]

Design a \( O(\log n) \) algorithm to find \( p \) given an array \( A \) with the above property.

4. Consider a complete binary tree \( T \) with \( n \) nodes (\( n = 2^d - 1 \) for some \( d \)). Each node \( v \) of \( T \) is labeled with a distinct real number \( x_v \). Define a node \( v \) of \( T \) to be a local minimum if the label \( x_v \) is less than the label \( x_w \) for all nodes \( w \) that are joined to \( v \) by an edge. You are given such a complete binary tree \( T \), but the labeling is only specified in the following implicit way: for each node \( v \), you can determine the value \( x_v \) by probing the node \( v \). Show how to find a local minimum of \( T \) using only \( O(\log n) \) probes to the nodes of \( T \).

5. Show that for any non-negative integers \( a, b \),

\[(n + a)^b = \Theta(n^b)\]

You need to show that \((n + a)^b\) is both \( O(n^b) \) as well as \( \Omega(n^b) \). Use the definitions of \( O(.) \) and \( \Omega(.) \).

Extra Credit: Show the above for the case when \( a \) is negative integer.