CMPS 102: Fall 2003
Homework Assignment 2

Due Monday 10/13/03

Prove each of the following statements by induction.

1. (10 Points) \( \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \) for all \( n \geq 1 \).

2. (10 Points) \( n^5 - n \) is divisible by 5 for all \( n \geq 0 \).

3. (20 Points) Define the function \( T(n) \) by the recurrence

\[
T(n) = \begin{cases} 
0 & \text{for } 1 \leq n < 3 \\
T(\lceil n/3 \rceil) + 2 & \text{for } n \geq 3 
\end{cases}
\]

Show that \( T(n) \leq 2 \log_3(n) \) for all \( n \geq 1 \). (Use strong induction.)

4. (20 Points) A graph is called complete if every pair of distinct vertices are joined by exactly one edge. Prove that for all \( n \geq 1 \), if \( G \) is a complete graph on \( n \) vertices, then \( G \) has exactly \( n(n-1)/2 \) edges.

5. (20 Points) Recall that a graph is called simple if it contains no loops or multiple edges, and is called connected if every pair of vertices are joined by at least one path. Prove that for all \( n \geq 1 \), if \( G \) is a simple connected graph on \( n \) vertices, then \( G \) has at least \( n-1 \) edges. (Use strong induction.)