Homework Assignment # 2

INDUCTION

Part A

Prove by induction:

1. \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n \geq 1 \).

2. \( \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1} \) for \( x \neq 1, n \geq 0 \).

3. Consider the following statement: the sum of cubes of the first \( n \) positive integers is equal to the square of the sum of these integers. Restate this as a formal mathematical theorem using \( \Sigma \)-notation. Prove your theorem.

4. \( n^5 - n \) is divisible by 5 for every positive integer \( n \).

5. Find (and prove) an exact closed form solution to \( f(n) \) mapping the natural numbers to the reals defined by

\[
f(n) = \begin{cases} 
  n & \text{if } n = 0 \text{ or } n = 1 \\
  5f(n - 1) + 6f(n - 2) & \text{otherwise}
\end{cases}
\]

You should first find a solution and then prove your answer by induction. (Hint: \( f(n) \) is a sum of exponentials like \( 3 \cdot 2^n + 5^n \). Look at small cases to guess the right sum).

6. Let \( F(0) = 1, F(1) = 6 \).
Let \( F(n) \) be defined by the following recurrence for \( n \geq 2 \):

\[
F(n) = 6F(n - 1) + 9F(n - 2).
\]

Using following steps for induction, prove that \( F(n) \geq n3^n \).
How many base cases do you have?
Prove the base case(s).
Write the inductive hypothesis very clearly.
Prove by induction.
How many times did you use the inductive hypothesis in your proof?
Part B

1. Prove the Binomial Theorem. Let \( n \) be a positive integer. Then,

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

2. Prove by structural induction that a complete binary tree with \( n \) levels has \( 2^n - 1 \) vertices.

Remark: I expect you to understand this material by October 4. There will be a quiz from Part A only. Solutions to Part B will not be posted. You are encouraged to write down the solutions to Part B, acknowledge all the help you get on Part B, write down the date when you did attempt part B, get a signature from a TA or a tutor within a day or two to verify that you did it on the date claimed by you and include it as a part of your portfolio at the end of the quarter.