QUIZ 1
CMPS 101 - Winter 02
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Name: __________________________
Student ID: ______________________

This exam is closed book and closed notes. Show all work. Partial credit given for partial solutions. Presentation counts! Be legible and coherent for full credit.

1. (25 points) Given the following recursive definition of \( \binom{n}{k} \):

\[
\binom{n}{k} = \begin{cases} 
1 & \text{if } k = n = 0, \\
\binom{n-1}{k-1} + \binom{n-1}{k} & \text{if } n > 0 \text{ and } 0 \leq k \leq n \\
0 & \text{if } k > n
\end{cases}
\]

Using this definition, prove the following for all nonnegative integers \( n \) and \( k \):

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

**Hint:** One method is to use induction on \( n \).
2. (25 points) Find the limit and simplify your answer.

\[
\lim_{{x \to \infty}} \frac{23^x}{3^x}
\]
3. (25 points)

- **Part 1** (12 points)

  Write C or Java code or pseudocode for “array_max,” a function with
  the following specifications:

  - has the C function declaration `int array_max(int a[][], int i, int j);`
    or the Java prototype `public long array_max(int a[][], int i, int j)`
    if you write code, or is a function that takes an i x j 2-dimensional
    array of integers and returns an integer if you write pseudocode.

  - `array_max(a[][], i, j)` returns the value of the largest int-
    ger in the array if every element in the array is positive.
    Otherwise, `array_max(a[][], i, j)` returns 0.
Name: 

• **Part 2** (13 points)
  Write a mathematical expression in terms of $i$ and $j$ for the number of comparisons performed by your algorithm for the function `array_max(a[], i, j)` you created in Part 1 above.
4. (25 points) Two very rich people play a gambling game. The first player has 100 cards numbered from 0 to 99 (exactly one card for each number). The first player shuffles the cards thoroughly so that you can assume every arrangement of cards is equally likely. The second player pays a stake of $1,000,000. Then the second player draws out one card. If \( n \) is the number of the card, the first player gives the second player a payoff of $100,000,000 multiplied by \( 10^{-n} \). For example if number on the card is 2, the payoff is $1,000,000, but if the number on the card is 9, the payoff is ten cents. Assume their accounting system is able to credit the fractional amounts when the payoff is less than one cent. If you consider this game only from the point of view of the expected value, is this game more favorable to the first player, more favorable to the second player, or equally favorable to each player? Show a mathematical calculation of the expected value. The final answer you compute need not be exact, you may give an approximation good enough to show whether the stake is less than, equal to or greater than the expected value of the payoff.

**Formula for expected value:**

\[
\mathcal{E}(X) = \sum_{u \in \text{domain of } X} u \cdot \Pr(X = u)
\]

**Hint:** First find the probability that the second player draws a particular card \( n \) from the deck. Every card is symmetric to every other card, and **nothing fancy is required**!! Then, calculate the payoff for card \( n \) and apply the formula for the expected value.
5. **Extra Credit**

In class, we gave the formula for the derivative with respect to $x$ of $b^x$ where $b$ is a constant: $\left[b^x\right]' = \ln b \cdot b^x$. Derive a formula for the derivative of the function

$$b^{[n]} = b^{(b^{(b^{(\cdots)}}))}$$

where $b^{[1]} = b^x$, $b^{[2]} = b^{(b^x)}$ and $b^{[n]}$ is a tower of $n$ such exponentiations of $b$, with $x$ as the exponent in the highest one. Prove your formula.