**Math Quiz**
Solve these problems on a separate sheet of paper. Show all work.

1. Simplify the summation.

\[
\frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{k}\right)
\]

We need to use some of the basic properties of summations plus the formula for the sum of an arithmetic series.

\[
\frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{k}\right)
= \frac{1}{n} \left(\sum_{i=1}^{n} 1 + \frac{1}{k} \sum_{i=1}^{n} i\right)
= \frac{1}{n} \left(n + \frac{1}{k} \cdot \frac{n(n+1)}{2}\right)
= 1 + \frac{1}{nk} \cdot \frac{n(n+1)}{2}
= 1 + \frac{n+1}{2k}
\]

**Note:** A summation similar to this comes up in Section 12.2, "Hash Tables," in Theorem 12.2 which gives the expected performance of retrieval in a hash table with chaining.
2. Find the limit.

\[
\lim_{x \to \infty} \frac{x}{\ln x \cdot \ln(\ln x)}
\]

I will give two methods, a direct way and a shortcut using a substitution.

**Method 1: Direct Solution**
We apply L'Hopital's rule repeatedly and simplify after each step.

\[
\lim_{x \to \infty} \frac{x}{\ln x \cdot \ln(\ln x)}
\]

\[
= \lim_{x \to \infty} \frac{1}{(1/x) \cdot \ln(\ln x) + (\ln x)(1/x)(1/\ln x)}
\]

\[
= \lim_{x \to \infty} \frac{1}{(1/x) \cdot \ln(\ln x) + (1/x)}
\]

\[
= \lim_{x \to \infty} \frac{1}{\ln(\ln x) + 1}
\]

\[
= \lim_{x \to \infty} \frac{x}{\ln(\ln x) + 1}
\]

\[
= \lim_{x \to \infty} \frac{1}{(1/x)(1/\ln x)}
\]

\[
= \lim_{x \to \infty} x \cdot \ln x = \infty
\]
**Method 2: Substitution**

We can simplify this problem significantly by performing a substitution to get rid of the product with nested logarithms, which create a lot of products and fractions with logs that make it so difficult to differentiate. We substitute a variable with exponentiation, because the exponential is very easy to differentiate.

Let \( u = \ln x \). Then \( x = e^u \) and \( \ln(\ln x) = \ln u \). Also, as \( x \to \infty \), \( u = \ln x \to \infty \), so the whole expression now becomes:

\[
\lim_{u \to \infty} \frac{e^u}{u \cdot \ln u}
\]

Applying L'Hôpital's rule repeatedly gives:

\[
\lim_{x \to \infty} \frac{e^u}{u \cdot 1/u + \ln u} = \lim_{x \to \infty} \frac{e^u}{1 + \ln u} = \lim_{x \to \infty} \frac{e^u}{1/u} = \lim_{x \to \infty} u \cdot e^u = \infty
\]

**Note:** We would perform such a calculation to show that \( \ln x \cdot \ln(\ln x) \) is in \( o(x) \). The function \( \ln x \cdot \ln(\ln x) \) appears sometimes in number theoretic algorithms related to cryptosystems (for example, see the Chapter Notes for Chapter 33, "Number Theoretic Algorithms," in the primary text).

3. Solve the indefinite integral.

\[
\int x \ln x \, dx
\]

This is a simple integration by parts. We want to get rid of the difficult \( \ln \) function, so we will choose to take the derivative of it and the antiderivative of \( x \). Thus, we let \( u = \ln x \) and \( dv = x \, dx \) to apply the integration method \( \int u \, dv = uv - \int v \, du \), which gives

\[
\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx
\]

\[
= \frac{x^2 \ln x}{2} - \int \frac{x}{2} \, dx
\]

\[
= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C
\]

To check the result, take its derivative to see that it returns to the original function.

\[
\left( \frac{x^2 \ln x - \frac{x^2}{4} + C \right)'
\]

\[
= x \ln x + x/2 - x/2 = x \ln x
\]

as required, so this answer is correct.
4. Prove that the sum of the first \( n \) odd numbers is \( n^2 \). This is a classic example of a mathematical induction proof. The key observation is to consider the algebraic identity, \((n + 1)^2 = n^2 + 2n + 1\) and note that \(2n + 1\) is the \( n + 1\)st odd number. We see that adding \(2n + 1\) to \(n^2\) gets us to \((n + 1)^2\), the next square number. But even if you didn’t see this idea immediately, you can still get the proof simply by setting up the induction framework and carrying out the algebraic steps. Following are the details of the proof.

The sum of the first \( n \) odd numbers is \( n^2 \). The proof is by induction on \( n \).

**Base Case:** The base case is the statement that the sum of the first odd number is 1. The first odd number is 1, and its sum is 1, so this is true.

**Inductive Step:** Note that the \( n \)th odd number is \(2n - 1\). The \( n + 1\)st odd number is then \(2n - 1 + 2 = 2n + 1\). We make the inductive hypothesis that the sum of the first \( n \) odd numbers equals \( n^2 \). To both sides of this equality we add \(2n + 1\). This gives us that the sum of the first \( n + 1 \) odd numbers equals \( n^2 + 2n + 1 \). By factoring, we find that the sum of the first \( n + 1 \) odd numbers is \((n + 1)^2\).

Some people prefer to follow the convention of taking an inductive step from \( n - 1 \) to \( n \) rather than from \( n \) to \( n + 1 \). We can also prove the inductive step in this way. We would use the inductive hypothesis that the sum of the first \( n - 1 \) odd numbers is \((n - 1)^2\), and the proof would continue as follows:

\[
\sum_{i=1}^{n-1} (2i - 1) = (n - 1)^2 \text{ by inductive hypothesis}
\]
\[
\sum_{i=1}^{n-1} (2i - 1) + 2n - 1 = (n - 1)^2 + 2n - 1 \text{ by adding } 2n - 1 \text{ to both sides}
\]
\[
\sum_{i=1}^{n} = n^2 - 2n + 1 + 2n - 1 \text{ by expansion}
\]
\[
\sum_{i=1}^{n} = n^2 \text{ by cancelling the opposite terms}
\]

For this class, I don’t care whether you prove the inductive step from \( n - 1 \) to \( n \) or from \( n \) to \( n + 1 \) as long as you are consistent and correct.

5. Suppose you have a coin box with two 10-cent coins and two 25-cent coins. You shake the box and three coins fall out. Assume that every coin is equally likely to fall out. What is the expected value of the three coins that fall out?

The random variable here is the value of the coins. The possible values it can take are 45 cents (two 10-cent coins and one 25-cent coin) or 60 cents (two 25-cent coins and one 10-cent coin).
We must compute the probability of these outcomes. Since the distributions of the coins are symmetric (two of each), the probability of getting two 10-cent coins and one 25-cent is the same as the probability of getting two 25-cent and one 10-cent. Thus, each of these outcomes has probability 1/2. Or, for those who prefer the full mathematical rigor, the probability of two 25-cents and one ten-cent is

\[
\frac{\binom{2}{2} \binom{2}{1}}{\binom{4}{3}} = \frac{2/4}{1/2} = 1/2
\]

and the probability of one 25-cent and two 10-cents is

\[
\frac{\binom{2}{1} \binom{2}{2}}{\binom{4}{3}} = 1/2
\]

which also equals 1/2, where

\[
\binom{n}{k}
\]

stands for the number of ways of choosing k out of n objects.

Now we can compute the expected value from the definition. Let X be a random variable representing the value of the coins. Then

\[
\mathcal{E}(X) = \sum_{u \in \{45, 60\}} u \cdot \Pr(X = u) = (1/2)45 + (1/2)60 = 105/2 = 52.5
\]