CMPS 101
Midterm 1 Review
Solutions to selected problems

1. State whether the following assertions are true or false. If any statements are false, give a related statement which is true.

   a. $f(n) = O(g(n))$ implies $f(n) = o(g(n))$.  **False**
      
      $f(n) = o(g(n))$ implies $f(n) = O(g(n))$
   
   b. $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.  **True**
   
   c. $f(n) = \Theta(g(n))$ if and only if $\lim_{n \to \infty} (f(n) / g(n)) = L$, where $0 < L < \infty$.  **False**
      
      $0 < L < \infty$ and $\lim_{n \to \infty} (f(n) / g(n)) = L$ implies $f(n) = \Theta(g(n))$

2. Prove that $\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$. In other words, if $h_1(n) = \Theta(f(n))$ and $h_2(n) = \Theta(g(n))$, then $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$.

   **Proof:**
   By hypothesis there exist positive constants $n_1, n_2, a_1, b_1, a_2,$ and $b_2$ such that
   
   $$\forall n \geq n_1 : 0 \leq a_1 f(n) \leq h_1(n) \leq b_1 f(n)$$

   and
   
   $$\forall n \geq n_2 : 0 \leq a_2 g(n) \leq h_2(n) \leq b_2 g(n)$$

   If $n \geq n_0 = \max(n_1, n_2)$, then both inequalities hold. Let $c = a_1 a_2$, and $d = b_1 b_2$. Since everything in sight is non-negative, we can multiply these two inequalities to get
   
   $$\forall n \geq n_0 : 0 \leq c f(n) g(n) \leq h_1(n) h_2(n) \leq d f(n) g(n),$$

   and hence $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$ as required.  ///

3. Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for all sufficiently large $n$), and suppose that $f(n) = \Theta(g(n))$. Does it necessarily follow that $\frac{1}{f(n)} = \Theta\left(\frac{1}{g(n)}\right)$? Either prove this statement, or give a counter-example.

   **Solution:**
   The statement is **true**, as we now prove. By hypothesis there exist positive numbers $c_1, c_2,$ and $n_0$ such that for all $n \geq n_0$: $0 < c_1 g(n) \leq f(n) \leq c_2 g(n)$. (Note: the strict inequality $<$ on the left follows from the fact that $f(n)$ and $g(n)$ are asymptotically positive.) Taking the reciprocals of all the positive terms in
this inequality gives: \( 0 < \frac{1}{c_2} \cdot \frac{1}{g(n)} \leq \frac{1}{f(n)} \leq \frac{1}{c_1} \cdot \frac{1}{g(n)} \) for all \( n \geq n_0 \). Observe that both \( \frac{1}{c_2} > 0 \) and \( \frac{1}{c_1} > 0 \), whence \( \frac{1}{f(n)} = \Theta\left(\frac{1}{g(n)}\right) \).

4. Give an example of functions \( f(n) \) and \( g(n) \) such that \( f(n) = o(g(n)) \) but \( \log(f(n)) \neq o(\log(g(n))) \).

(Hint: Consider \( n! \) and \( n^n \) and use the corollary to Stirling’s formula in the handout on common functions.)

Solution:
Following the hint, we let \( f(n) = n! \) and \( g(n) = n^n \). Part (1) of the Corollary to Stirling’s formula on page 3 of the handout on common functions showed that \( f(n) = o(g(n)) \). Part (3) of that same Corollary showed that \( \log(n!) = \Theta(n \log(n)) \), and hence \( \log(f(n)) = \Theta(n \log(n)) = \Theta(\log(n^n)) = \Theta(\log(g(n)) \). Since \( o(\log(g(n)) \cap \Theta(\log(g(n)) = \emptyset \) by problem 6 below, we have \( \log(f(n)) \neq o(\log(g(n))) \).

5. Let \( g(n) \) be an asymptotically non-negative function. Prove that \( o((g(n)) \cap \Omega(g(n)) = \emptyset \).

Proof:
Assume to get a contradiction that \( f(n) \in o((g(n)) \cap \Omega(g(n)) \). Then since \( f(n) = \Omega(g(n)) \) we have

\( (1) \quad \exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1: 0 \leq c_1 g(n) \leq f(n) \)

Also, since \( f(n) = o(g(n)) \) we have

\( (2) \quad \forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2: 0 \leq f(n) < c_2 g(n) \)

Let \( c_2 = c_1 \). Then \( c_2 > 0 \), and by (2) there exists \( n_2 > 0 \) such that \( 0 \leq f(n) < c_1 g(n) \) for all \( n \geq n_2 \). Pick any \( m \geq \max(n_1, n_2) \). Then by (1) we have \( 0 \leq c_1 g(m) \leq f(m) < c_1 g(m) \), and hence \( c_1 g(m) < c_1 g(m) \), a contradiction. Our assumption was therefore false, and no such function \( f(n) \) can exist. We conclude that \( o((g(n)) \cap \Omega(g(n)) = \emptyset \).
7. (d) Use limits to prove the following: \( f(n) + o(f(n)) = \Theta(f(n)) \)

**Proof:**

In this equation, the term \( o(f(n)) \) stands for some function \( h(n) \) satisfying \( \lim_{n \to \infty} \left( \frac{h(n)}{f(n)} \right) = 0 \). Therefore

\[
\lim_{n \to \infty} \left( \frac{f(n) + h(n)}{f(n)} \right) = \lim_{n \to \infty} \left( 1 + \frac{h(n)}{f(n)} \right) = 1 + \lim_{n \to \infty} \left( \frac{h(n)}{f(n)} \right) = 1,
\]
showing that \( f(n) + h(n) = \Theta(f(n)) \). Note that this result justifies the practice of dropping low order terms when finding the asymptotic growth rate of a function.

8. Let \( g(n) = n \) and \( f(n) = n + \frac{1}{2} n^2 (\sin(n) + 1) \). Show that

a. \( f(n) = \Omega(g(n)) \)

b. \( f(n) \neq O(g(n)) \)

c. \( \lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) \) does not exist, even in the sense of being infinite.

**Note:** this is the ‘Example C’ mentioned in the handout on asymptotic growth rates.

**Proof of (a):**

For any \( n \geq 1 \) we have \(-1 \leq \sin(n) \leq 1\) and hence \( \sin(n) + 1 \geq 0 \). Thus

\[
f(n) = n + \frac{1}{2} n^2 (\sin(n) + 1) \geq n = g(n).
\]

Thus \( 0 \leq 1 \cdot g(n) \leq f(n) \) for all \( n \geq 1 \), whence \( f(n) = \Omega(g(n)) \).

**Proof of (b):**

We must show that the sentence \( \exists c > 0, \forall n_0 > 0, \forall n \geq n_0: 0 \leq f(n) \leq c \cdot g(n) \) is false. We do this by showing that its negation \( \forall c > 0, \forall n_0 > 0, \exists n \geq n_0: c \cdot g(n) < f(n) \) is true. Pick \( c > 0 \) and \( n_0 > 0 \) arbitrarily. Define \( n = \frac{\pi}{2} + 2\pi \cdot k \) where the integer \( k \) is chosen so large as to guarantee that \( n \geq \max(c, n_0) \).

(This is possible since \( \frac{\pi}{2} + 2\pi \cdot k \to \infty \) as \( k \to \infty \).) Then \( n \geq n_0 \) and \( n \geq c > c - 1 \), whence \( n + 1 > c \).

Observe also that \( \sin(n) = 1 \), and therefore

\[
f(n) = n + \frac{1}{2} n^2 (\sin(n) + 1) = n + n^2 = n(1 + n) > cn = c \cdot g(n)
\]
as required.
Proof of (c):
Observe that
\[
\frac{f(n)}{g(n)} = \frac{n + \frac{1}{2} n^2 (\sin(n) + 1)}{n} = 1 + \frac{1}{2} n (\sin(n) + 1),
\]
which oscillates with increasing amplitude between 1 and 1 + n as \( n \to \infty \), and therefore has no limit, even in the sense of being infinite. / / /

10. Use Stirling’s formula to prove that \( \binom{2n}{n} = \Theta \left( \frac{4^n}{\sqrt{n}} \right) \).

Proof:
By Stirling’s formula
\[
\binom{2n}{n} = \frac{(2n)!}{n! (2n - n)!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{2\pi \cdot 2n} \cdot \left( \frac{2n}{e} \right)^{2n} \cdot (1 + \Theta(1/2n))}{\sqrt{2\pi n} \cdot \left( \frac{n}{e} \right)^n \cdot (1 + \Theta(1/n))^2} = \frac{2^{2n} \cdot 1 + \Theta(1/2n)}{\sqrt{\pi n} \cdot (1 + \Theta(1/n))^2} = \frac{1}{\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \cdot \frac{1 + \Theta(1/2n)}{(1 + \Theta(1/n))^2}
\]
so that
\[
\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1 + \Theta(1/2n)}{(1 + \Theta(1/n))^2} \to \frac{1}{\sqrt{\pi}} \quad \text{as} \quad n \to \infty
\]
The result now follows since \( 0 < \frac{1}{\sqrt{\pi}} < \infty \). / / /

11. Consider the following sketch of an algorithm called ProcessArray which performs some unspecified operation on a subarray \( A[p \cdots r] \).

\[
\text{ProcessArray}(A, p, r) \quad \text{(Preconditions: } 1 \leq p \text{ and } r \leq \text{length}[A])
\]
1. Perform 1 basic operation
2. if \( p < r \)
3. \( q \leftarrow \left\lfloor \frac{p + r}{2} \right\rfloor \)
4. ProcessArray(A, p, q)
5. ProcessArray(A, q+1, r)
a. Write a recurrence formula for the number $T(n)$ of basic operations performed by this algorithm when called on the full array $A[1\cdots n]$, i.e. by ProcessArray($A$, 1, $n$). (Hint: recall our analysis of MergeSort.)

Solution:

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 1 & n \geq 2 \end{cases}$$

b. Show that the solution to this recurrence is $T(n) = 2n - 1$, whence $T(n) = \Theta(n)$.

Proof:

Observe that when $n = 1$ we have $T(1) = 2 \cdot 1 - 1 = 1$. When $n \geq 2$ we have

$$\text{RHS} = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 1$$
$$= (2\lceil n/2 \rceil - 1) + (2\lfloor n/2 \rfloor - 1) + 1$$
$$= 2(\lceil n/2 \rceil + \lfloor n/2 \rfloor) - 1$$
$$= 2n - 1$$
$$= T(n)$$
$$= \text{LHS}$$

12. Consider the following algorithm which does nothing but waste time:

\begin{algorithm}
\begin{enumerate}
\item \textbf{WasteTime}($n$) \hspace{1em} (pre: $n \geq 1$)
\item if $n > 1$
\item for $i \leftarrow 1$ to $n^3$
\item waste 2 units of time
\item for $i \leftarrow 1$ to 7
\item WasteTime($\lceil n/2 \rceil$)
\item waste 3 units of time
\end{enumerate}
\end{algorithm}

a. Write a recurrence formula for the amount of time $T(n)$ wasted by this algorithm.

Solution:

$$T(n) = \begin{cases} 0 & n = 1 \\ 7T(\lceil n/2 \rceil) + 2n^3 + 3 & n \geq 2 \end{cases}$$

b. Show that when $n$ is an exact power of 2, the solution to this recurrence relation is given by $T(n) = 16n^3 - \frac{1}{2} - \frac{31}{2}n^{\log_2 7}$, and hence $T(n) = \Theta(n^3)$. 
Proof:
If $n=1$ then $T(1)=16 \cdot 1^3 - \frac{1}{2} - \frac{31}{2} \log_7 1 = 16 - \frac{32}{2} = 0$. When $n \geq 2$ is an exact power of 2 we have

$$\text{RHS} = 7T(n/2) + 2n^3 + 3$$

$$= 7\left(16\left(\frac{n}{2}\right)^3 - \frac{1}{2} - \frac{31}{2} \left(\frac{n}{2}\right)^{\log_7 2}\right) + 2n^3 + 3$$

$$= 7\left(16\left(\frac{n}{8}\right)^3 - \frac{1}{2} - \frac{31}{2} \left(\frac{n}{7}\right)^{\log_7 2}\right) + 2n^3 + 3$$

$$= 14n^3 - \frac{7}{2} - \frac{31}{2} n^{\log_7 2} + 2n^3 + \frac{6}{2}$$

$$= 16n^3 - \frac{1}{2} - \frac{31}{2} n^{\log_7 2}$$

$$= T(n)$$

$$= \text{LHS}$$

13. Define $T(n)$ by the recurrence formula

$$T(n) = \begin{cases} 
1 & 1 \leq n < 3 \\
2T(\lfloor n/3 \rfloor) + 4n & n \geq 3 
\end{cases}$$

Use Induction to show that $\forall n \geq 1 : T(n) \leq 12n$, and hence $T(n) = O(n)$.

Proof: (Multiple base cases, strong version)
I. Observe $T(1)=1 \leq 12 \cdot 1$ and $T(2)=1 \leq 12 \cdot 2$, so the base cases are satisfied.
IId. Let $n \geq 3$ and assume for all $k$ in the range $1 \leq k < n$ that $T(k) \leq 12k$. In particular, since $1 \leq \lfloor n/3 \rfloor < n$, we have $T(\lfloor n/3 \rfloor) \leq 12 \lfloor n/3 \rfloor$. We must show that $T(n) \leq 12n$. Observe

$$T(n) = 2T(\lfloor n/3 \rfloor) + 4n \quad \text{by the recurrence formula for } T(n)$$

$$\leq 2 \cdot 12 \lfloor n/3 \rfloor + 4n \quad \text{by the induction hypothesis}$$

$$\leq 2 \cdot 12(n/3) + 4n \quad \text{since } \lfloor x \rfloor \leq x \text{ for any real number } x$$

$$= 8n + 4n$$

$$= 12n$$

The result now holds for all $n \geq 3$.  

///
15. Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence:

$$S(n) = \begin{cases} 
0 & \text{if } n = 1 \\
S\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 1 & \text{if } n \geq 2 
\end{cases}$$

Use induction to prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

**Proof:** Let $P(n)$ be the inequality $S(n) \geq \lg(n)$.

I. The inequality $S(1) \geq \lg(1)$ reduces to $0 \geq 0$, which is obviously true, so $P(1)$ holds.

II. Let $n > 1$ and assume for all $k$ in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. Then

$$S(n) = S\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 1 \quad \text{by the definition of } S(n)$$

$$\geq \lg\left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \text{by the induction hypothesis with } k = \left\lfloor \frac{n}{2} \right\rfloor$$

$$\geq \lg(n/2) + 1 \quad \text{since } \left\lfloor x \right\rfloor \geq x \text{ for any } x$$

$$= \lg(n) - \lg(2) + 1$$

$$= \lg(n)$$

showing that $P(n)$ holds. Therefore $S(n) \geq \lg(n)$ for all $n \geq 1$, as claimed. // /

16. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i) = \Theta(nf(n))$$

(Hint: Emulate the Example on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ for any positive integer $k$)

**Proof:**

Since $f(n)$ is increasing we have $\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} f(n) = nf(n) = O(nf(n))$. Note also that

$$\sum_{i=1}^{n} f(i) \geq \sum_{i=\lceil n/2 \rceil}^{n} f(i) \quad \text{by discarding some positive terms}$$

$$\geq \sum_{i=\lceil n/2 \rceil}^{\lceil n/2 \rceil} f(\lceil n/2 \rceil) \quad \text{since } f(n) \text{ is increasing}$$

$$= (n - \left\lceil n/2 \right\rceil + 1)f(\left\lceil n/2 \right\rceil) \quad \text{by counting terms}$$

$$= (\left\lceil n/2 \right\rceil + 1)f(\left\lceil n/2 \right\rceil) \quad \text{since } n = \left\lfloor n/2 \right\rfloor + \left\lceil n/2 \right\rceil$$

$$> (\left\lfloor n/2 \right\rfloor - 1 + 1)f(n/2) \quad \text{since } f(n) \text{ is increasing, } \left\lfloor x \right\rfloor \geq x, \text{ and } \left\lfloor x \right\rfloor > x - 1$$

$$= (n/2)f(n/2)$$

$$= \Omega(nf(n)) \quad \text{since } f(n/2) = \Theta(f(n)), \text{ whence } f(n/2) = \Omega(f(n))$$

It follows from an exercise in the handout on Asymptotic Growth rates that $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$, as claimed. // /
17. Use the result of the preceding problem to give an alternate proof of \( \log(n!) = \Theta(n \log(n)) \) that does not use Stirling’s formula.

**Proof:**
Observe that \( \log(n) \) is a positive increasing function, and that \( \log(n/2) = \log(n) - \log(2) = \Theta(\log(n)) \). We may therefore apply the result of problem 17 with \( f(n) = \log(n) \), and properties of logarithms to get

\[
\log(n!) = \sum_{i=1}^{n} \log(i) = \Theta(n \log(n))
\]

as claimed. // /

18. Let \( T(n) \) be defined by the recurrence formula

\[
T(n) = \begin{cases} 
1 & n = 1 \\
T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 
\end{cases}
\]

Show that \( \forall n \geq 1: T(n) \leq \frac{4}{3} n^2 \), and hence \( T(n) = O(n^2) \).

**Proof:** Let \( P(n) \) be the statement \( T(n) \leq (4/3)n^2 \). Then \( P(1) \) is true, since \( T(1) = 1 \leq 4/3 = (4/3) \cdot 1^2 \), and the base case is satisfied. Let \( n > 1 \) be chosen arbitrarily, and suppose for all \( k \) in the range \( 1 \leq k < n \) that \( T(k) \leq (4/3)k^2 \). We must show as a consequence that \( T(n) \leq (4/3)n^2 \). Observe

\[
T(n) = T(\lfloor n/2 \rfloor) + n^2 \quad \text{by the recurrence formula for } T(n)
\]

\[
\leq (4/3) \lfloor n/2 \rfloor^2 + n^2 \quad \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor
\]

\[
\leq (4/3)(n/2)^2 + n^2 \quad \text{since } \lfloor x \rfloor \leq x \text{ for any } x
\]

\[
= n^2/3 + n^2 = n^2/3 + n^2 = (4/3)n^2,
\]

as required. // /