B.4 Graphs

A directed graph is a pair $G = (V, E)$ of sets where $V$ is finite and non-empty (called vertices) and $E \subseteq V \times V$, i.e. $E$ consists of ordered pairs of vertices (called edges).

In an undirected graph the edge set $E$ consists of unordered pairs of vertices.

**Example:**

**Directed**

\[
\begin{array}{c}
\text{x} \\
\text{u} \\
\text{v} \\
\text{y} \\
\end{array}
\]

\[V = \{x, y, u, v\} \]

\[E = \{(x, y), (u, x), (y, v), (v, u), (x, u)\}\]

**Undirected**

\[
\begin{array}{c}
\text{x} \\
\text{u} \\
\text{v} \\
\text{y} \\
\end{array}
\]

\[V = \{x, y, u, v\} \]

\[E = \{xy, xu, yv, uy, yu\}\]

In the above example we would say $x$ is adjacent to $y$, $x$ is incident with $xy$, and $xy$ is adjacent to $xu$.

The degree of a vertex is the number of edges incident with it. In a directed graph we have obvious notions of indegree and outdegree.
An **x-y path** is a sequence of vertices

\[ P: x = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = y \]

for which \((v_{i-1}, v_i) \in E\) for \(1 \leq i \leq k\). The length of such a path is the number of edges \(k\).

We call \(x = v_0\) the initial vertex, \(y = v_k\) the terminal vertex, and \(v_1, \ldots, v_{k-1}\) the intermediate or internal vertices. A path \(P\) is called simple if its internal vertices are distinct.

\(P\) is called a cycle if its initial and terminal vertices are identical (i.e., \(v_0 = v_k\)).

In a directed graph we have obvious notions of directed \& undirected paths, cycles, etc.

An undirected graph \(G\) is called connected if for all \(x, y \in V(G)\), \(G\) contains an \(x-y\) path. A directed graph is called connected if its underlying undirected graph is connected. A directed graph is called *strongly connected* if every vertex is reachable from every other vertex along a directed path.
A graph $G$ is called **acyclic** (also a **forest** if it contains no cycle).

A **tree** is a graph which is both acyclic and connected.

Connected | Disconnected
---|---

Strongly connected | Not strongly connected

Forest

Tree
22.1 Graph Representations

Let $G$ be a graph (undirected) and $x, y \in V(G)$. The distance $d(x, y)$ from $x$ to $y$ is defined as

$$d(x, y) = \begin{cases} 
\text{length of a shortest } x-y \text{ path} \\
\infty \text{ if no such path exists.}
\end{cases}$$

Single Source Shortest Path Problem (SSSP):

Given $s \in V(G)$, determine $d(s, y)$ for all $y \in V(G)$. As we'll see, this problem is solved by the Breadth First Search (BFS) algorithm.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of $G$ is the $n \times n$ matrix $A = A(G)$ given by

$$A_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \in E \\
0 & \text{if } (v_i, v_j) \not\in E
\end{cases}$$

Ex

$$A = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}$$
Remarks

(1) \( A(G) \) is necessarily symmetric.

(2) The sum of the \( i \)th row (or column) of \( A(G) \) is \( \deg(v_i) \).

(3) Handshake Lemma

\[
\sum_{i=1}^{n} \deg(v_i) = 2|E(G)|
\]

(4) The \( i \)th row \( j \)th column of \( A^d \) is the number of \( v_i - v_j \) paths in \( G \) of length \( d \).

Ex.

\[
A = \begin{pmatrix}
3 & 1 & 1 & 2 \\
1 & 2 & 2 & 1 \\
1 & 2 & 2 & 1 \\
2 & 1 & 1 & 3 \\
\end{pmatrix}
\]

The adjacency list representation of \( G \) consists of an array \( \text{Adj}[i] \) of \( n = |V(G)| \) lists. \( \text{Adj}[i] \) contains the neighbors of vertex \( v_i \).

Ex.

\[
\begin{align*}
1 & : 2, 3, 4 \\
2 & : 1, 4 \\
3 & : 1, 4 \\
4 & : 1, 2, 3
\end{align*}
\]
Using our doubly linked lists of Pa1 and Pa2, we have:

![Diagram of doubly linked lists and array]

Array of List References

The order in which vertices are stored in Adj[i][j] may be arbitrary, but the particular order will affect the operations of BFS (and other algorithms which use ALR.)

Note: The adjacency matrix and adjacency list representations are valid on directed graphs. $A(G)$ may not be symmetric. The definitions of $Sl(x,y)$ and $SSSP$ go through unchanged.

Ex. $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

\[A \begin{pmatrix} \text{1} \\ \text{2} \\ \text{3} \\ \text{4} \end{pmatrix} = \begin{pmatrix} \text{2} \\ \text{4} \\ \text{1} \\ \text{3} \end{pmatrix}\]