1. (20 Points) Determine whether the following statements are true or false. Prove or disprove each statement accordingly.

   a. (10 Points) If $a < b$, then $a^n = o(b^n)$.  **True**

   **Proof:**
   Since $a/b < 1$ we have $\lim_{n \to \infty} \left(\frac{a^n}{b^n}\right) = \lim_{n \to \infty} \left(\frac{a}{b}\right)^n = 0$, whence $a^n = o(b^n)$.  ■

   b. (10 Points) If $a > b$, then $a^{\log_b(n)} = \omega(n)$.  **True**

   **Proof:**
   Since $a > b$, we have $\log_b(a) > \log_b(b) = 1$, and $\log_b(a) - 1 > 0$. Therefore
   \[
   \lim_{n \to \infty} \left(\frac{a^{\log_b(n)}}{n}\right) = \lim_{n \to \infty} \left(\frac{n^{\log_b(a)}}{n}\right) = \lim_{n \to \infty} \left(n^{\log_b(a) - 1}\right) = \infty,
   \]
   and hence $a^{\log_b(n)} = \omega(n)$.  ■

2. (20 Points) Use Stirling’s formula to prove that $\log(n!) = \Theta(n \log(n))$.

   **Proof:**
   Taking log (any base) of both sides of Stirling's formula, and using the laws of logarithms, we get
   \[
   \log(n!) = \log \left(\sqrt{2\pi n} \cdot (n/e)^n \cdot (1 + \Theta(1/n))\right) = \log \sqrt{2\pi n} + \log(n/e)^n + \log(1 + \Theta(1/n))
   \]
   \[
   = \frac{1}{2} \log(2\pi) + \frac{1}{2} \log(n) + n \log(n) - n \log(e) + \log(1 + \Theta(1/n)).
   \]
   Therefore
   \[
   \frac{\log(n!)}{n \log(n)} = \frac{\log(2\pi)}{2n \log(n)} + \frac{1}{2n} + 1 - \frac{\log(e)}{\log(n)} + \frac{\log(1 + \Theta(1/n))}{n \log(n)},
   \]
   and hence
   \[
   \lim_{n \to \infty} \left(\frac{\log(n!)}{n \log(n)}\right) = 1.
   \]
   Thus $\log(n!) = \Theta(n \log(n))$, as claimed.  ■
3. (20 Points) Consider the following algorithm that wastes time.

\[
\text{WasteTime}(n) \quad \text{(pre: } n \geq 1) \\
1. \quad \text{if } n = 1 \\
2. \quad \text{waste 2 units of time} \\
3. \quad \text{else} \\
4. \quad \text{WasteTime}([n/2]) \\
5. \quad \text{WasteTime}([n/2]) \\
6. \quad \text{waste 5 units of time}
\]

a. (10 Points) Write a recurrence relation for the number of units of time \( T(n) \) wasted by this algorithm.

Solution:

\[
T(n) = \begin{cases} 
2 & n = 1 \\
T([n/2]) + T([n/2]) + 5 & n \geq 2
\end{cases}
\]

b. (10 Points) Show that \( T(n) = 7n - 5 \) is the solution to this recurrence. (Hint: you may use without proof the fact that \( [n/2] + [n/2] = n \).)

Proof:
First observe that if \( T(n) = 7n - 5 \), then \( T(1) = 7 - 5 = 2 \). Second, if \( n \geq 2 \) we have

\[
\text{RHS} = T([n/2]) + T([n/2]) + 5 \\
= (7[n/2] - 5) + (7[n/2] - 5) + 5 \\
= 7([n/2] + [n/2]) - 5 \\
= 7n - 5 = T(n) = \text{LHS},
\]

showing that \( T(n) = 7n - 5 \) solves the recurrence.
4. (20 Points) Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T([n/2]) + n^2 & n \geq 2 \end{cases}$$

a. (4 Points) Determine the values $T(2), T(3), T(4)$ and $T(5)$.

**Solution:**

- $T(2) = T(1) + 2^2 = 1 + 4 = 5$
- $T(3) = T(1) + 3^2 = 1 + 9 = 10$
- $T(4) = T(2) + 4^2 = 5 + 16 = 21$
- $T(5) = T(2) + 5^2 = 5 + 25 = 30$

b. (16 Points) Prove that $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. (Hint: use strong induction.)

**Proof:**

**Base Step**

Observe that $T(1) = 1 \leq \frac{4}{3} \cdot 1^2$, which establishes the base case.

**Induction Step (IId)**

Let $n > 1$ be chosen arbitrarily. Assume for all $k$ in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. We must show as a consequence that $T(n) \leq (4/3)n^2$. Observe

$$T(n) = T([n/2]) + n^2$$

by the recurrence formula for $T(n)$

$$\leq (4/3)[n/2]^2 + n^2$$

by the induction hypothesis with $k = [n/2]$

$$\leq (4/3)(n/2)^2 + n^2$$

since $[x] \leq x$ for any $x$

$$= n^2 / 3 + n^2$$

as required. It follows from the second principle of mathematical induction that $T(n) \leq \frac{4}{3}n^2$ for all $n \geq 1$. □
5. (20 Points) Prove that for all $n \geq 1$: if $T$ is a tree on $n$ vertices, then $T$ has $n - 1$ edges. (Hint: you may use the following fact without proof: removing an edge from a tree results in exactly two trees, each with fewer vertices than the original.)

**Proof:**

I. There is only one tree with a single vertex and it has no edges. Therefore if $T$ is a tree on 1 vertex, then $T$ has $0 = 1 - 1$ edges. The base case is therefore satisfied.

II. Let $n > 1$ be chosen arbitrarily, and assume for all $k$ in the range $1 \leq k < n$ that if $T'$ is a tree on $k$ vertices, then $T'$ has $k - 1$ edges. We must show that if $T$ is a tree on $n$ vertices, then $T$ has $n - 1$ edges.

Assume $T$ is a tree on $n$ vertices. Pick any edge $e \in E(T)$ and remove it. By the above hint, this removal results in two trees $T_1$ and $T_2$, each with fewer than $n$ vertices. Say $T_1$ has $k_1 < n$ vertices and $T_2$ has $k_2 < n$ vertices. By the induction hypothesis $T_1$ has $k_1 - 1$ edges and $T_2$ has $k_2 - 1$ edges. Since no vertices were removed, we also have $k_1 + k_2 = n$. Therefore the number of edges in our original tree $T$ must be

\[
(# \text{ of edges in } T_1) + (# \text{ of edges in } T_2) + 1 = (k_1 - 1) + (k_2 - 1) + 1 \\
= (k_1 + k_2) - 1 \\
= n - 1,
\]

as required. The result follows for all $n \geq 1$ by the 2nd Principle of Mathematical Induction. ■