1. Part 1 ext. 1 more day

2. Runtime of DFS:
   \[ n = |V|, \quad m = |E| \]
   - Loop 1-3: \( \Theta(n) \)
   - Loop 5-7 (disregarding line 7): \( \Theta(n) \)
   - DFS, 7 and Visit. 6 (calls to Visit) are executed \( n \) times,

   - Visit: Loop 3-8 (excluding 6)
     is executed \( |\text{adj}(x)| \) times

   \[ \sum_{x \in V} |\text{adj}(x)| = \begin{cases} m & \text{directed} \\ 2m & \text{undirected} \end{cases} = \Theta(m) \]
So total cost of all calls to Visit() is $\Theta(m)$.

Total cost of DFS: $\Theta(n+m)$

What structural information about $G$ is stored in PL1 and in PL2?

Note: PL1 and PL2 store the same information.
Parenthesis string: "(" discover time "\)")" finish time

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Theorem

Let \( x, y \in V \), let \( d[x] < d[y] \). Then exactly one of the following is true.
(1) $d[x] < d[y] < d[y] < d[y]$

\[
\begin{array}{c}
\downarrow \\
\text{x} \quad \text{y}
\end{array}
\]

or

(2) $d[x] < d[y] < d[y] < d[x]$

\[
\begin{array}{c}
\downarrow \\
\text{y} \\
\downarrow \\
\text{x}
\end{array}
\]

Ranks

(2) is equiv. to saying that $y$ is discovered while $x$ is gray, which is equiv. to $y$ is a descendant of $x$ in some tree in the DFS forest.
• (1) holds therefore when y is not a descendant of x: either x, y are cousins in same tree, or x, y lie in different trees.

Proof: see book.

Theorem (White Path)

Let x, y ∈ V. Then y is a descendant of x iff at time d[x], G contains an x-y path consisting of white vertices.

Proof: see book.
classification of edges

(1) Tree edges: Belong to $G_p$, the DFS forest.

(2) Back edges: Join a vertex to an ancestor.

(3) Forward edges: Join a vertex to a descendant (other than a child).

(4) Cross edges: Join tree to tree, or cousin to cousin.

Note: Distinction between (2) and (3) only makes sense for Digraphs. In an undirected graph there is only one category: Back edges.
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Tree: (1, 2), (2, 3), (3, 4), (5, 6), (5, 7)
Back: (3, 1)
Forward: (1, 4)
Cross: (5, 2), (6, 4)  true-truetrue
(7, 6)  cousin-cousin

Theorem

If G is an undirected graph, then after DFS(G), there are no cross edges.

Proof. See book.
Exercise
modify DFS so it prints out the classification of each edge as it runs. (See pp. 546–548 and Prob. 22.3–4)

Exercise
modify DFS so that it finds the connected components of an undirected graph.
23.4 Topological Sort

**Definition**
A directed graph $G$ is called **acyclic** if $G$ contains no directed cycles.

Called: Directed Acyclic Graph (DAG)

**Lemma**
A directed graph $G$ is acyclic iff $DFS(G)$ yields no back edges.
Ex

\[ \begin{array}{cc}
\text{\shortstack{\rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow}} & \text{\shortstack{\rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow}} \\
\text{DAC} & \text{not a DAC}
\end{array} \]

Note: Lemma is logically equivalent to

Corollary

A digraph contains a directed cycle \( \iff \) DFS produces a back edge.
\[ \text{Proof} \]

\[ (\Leftarrow) \text{ obvious.} \]

\[ (\Rightarrow) \text{ (uses White Path Theorem).} \]

Suppose \( C \) contains a digraph cycle \( C \).

Let \( y \) be the first vertex on \( C \) to be discovered by DFS.
let $x$ be the predecessor of $y$ along this cycle. Note at time $d[y,x]$ the path from $y$ to $x$ along this cycle is all white.

By the white path theorem, $x$ must be a descendant of $y$ in some DFS tree. So $(x,y)$ is a back edge.