continue. Ex 4 in ind. handout!

For $n \geq 2$, $\mathcal{H}(1 \ldots n) \implies \mathcal{H}(n)$.

Let $n \geq 2$ be arbitrary. Assume for all $k$ in range $1 \leq k < n$ that $T(k) \leq k^2$.

In particular, for $k = \lfloor \frac{n}{3} \rfloor$, we have

$$T(\lfloor \frac{n}{3} \rfloor) \leq \left( \frac{n}{3} \right)^2.$$  

We must show: $T(n) \leq n^2$.

So

$$T(n) = 4 \cdot T(\lfloor \frac{n}{3} \rfloor) + n$$

$$\leq 4 \cdot \left( \frac{n}{3} \right)^2 + n \quad (\text{by the})$$

$$\quad (\text{ind. hyp.})$$
\[
\leq 4 \cdot \left( \frac{n}{3} \right)^2 + n \quad \text{(since } 1 \times 1 \leq x) .
\]
\[
= \frac{4}{9} n^2 + n \leq n^2 \\
\uparrow
\text{ want}
\]

Last inequality follows from

\[
n > 2 \implies \frac{9}{5} \leq n
\]
\[
\implies n \leq \frac{5}{9} n^2
\]
\[
\implies \frac{4}{9} n^2 + n \leq n^2
\]

\[\text{By 2nd PMA: } \overline{T}(n) \leq n^2 \]

for all \( n \geq 1 \).
A graph is a pair of sets $G = (V, E)$ where

$V 
eq \emptyset$: vertex set

$E \subseteq \binom{V}{2}$: edge set

2-subsets $\{x, y\} = xy = yx$ of $V$

Examples:

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{12, 14, 23, 24, 25, 26, 35, 36, 45, 56\}$

edge 14 is adjacent to 14

3 is adjacent to 6

3 is incident with 36
Let \( x, y \in V(G) \), an \( x \rightarrow y \) path in a sequence of vertices

\[ x = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = y \]

such that (1) \( v_{i-1}, v_i \in E(G) \)

for all \( 1 \leq i \leq k \), (2) no vertices are repeated (except possibly when \( x = y \)).

If \( x = y \) we speak of a \underline{closed path}, a \underline{closed path} with at least 2 edges is a \underline{cycle}.
1-6 path: 1, 2, 4, 5, 3, 6

1-6 path: 1, 2, 6

cycle: 2, 3, 6, 5, 2

Trivial path: 4

Note:

\[ x \quad \circ \quad \circ \quad \circ \quad y \]

not allowed

\[ x \quad \circ \quad \circ \quad \circ \quad \circ \quad x \]
A graph is connected if for all \( x, y \in V(G) \), \( G \) contains an \( x-y \) path. Otherwise \( G \) is said to be disconnected.

Example:

\[
\begin{array}{c}
1 & 2 & 3 & 4 \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
5 & 6 & 7 & 8 & 9
\end{array}
\]

Define a subgraph of \( G = (V, E) \) is a graph \( H = (V', E') \) s.t. \( V' \subseteq V \), and \( E' \subseteq E \).
Ex.

- \( \{1, 2, 5\}, \{12, 15, 25\} \)
  - connected

- \( \{2, 3, 4, 7\}, \{26, 37\} \)
  - disconnected

- \( \{2, 3, 9\}, \{49\} \)
  - not a graph, so not a subgraph
A subgraph $H$ of $G$ is called connected component of $G$ iff

1. $H$ is connected
2. $H$ is maximal w.r.t. (1)

Ex. conn. comps

- $(2, 1, 2, 5, 6)$, $\{12, 15, 25, 26, 567\}$
- $(2, 3, 7, 8)$, $\{37, 38, 789\}$
- $(4, 97, 494)$
Note: $G$ is connected iff it has exactly 1 component.

**Defn**

A graph $G$ is called **acyclic** iff it contains no cycles. (also called a forest.)

**Defn**

$G$ is a **tree** iff it is both acyclic and connected.
Ex. Acyclic

Vertex: 7 6 5
Edges: 6 5 4

Theorem
For all $n \geq 1$: if $T$ is a tree with $n$ vertices, then $T$ has $n-1$ edges.
1. If $n=1$, then $T$ can have no edges, so \# of edges $= 0 = n-1$.

   - Only tree with 1 vertex.

2. Let $n > 1$. Assume for all $k$ in the range $1 \leq k < n$ that if $T'$ is a tree on $k$ vertices, then $T'$ has $k-1$ edges.
we must show that if T is a tree on n vertices, then T has n-1 edges.

Suppose T is a tree on n vertices. Pick any edge e \in E(T) and remove it.

This results in two subtrees: T_1 and T_2, each with fewer than n vertices.

Say T_i has k_i vertices (i=1, 2). Then each k_i < n.
By the induction hypothesis, \( T_i \) has \( k_i - 1 \) edges (\( i = 1, 2 \)).

Note: \( k_1 + k_2 = n \) since no vertices were removed. By replacing e into \( T \) we see that the number of edges in \( T \) is

\[
(k_1 - 1) + (k_2 - 1) + 1
\]

\[= (k_1 + k_2) - 1\]

\[= n - 1 \text{, as required.}\]

Result holds for all trees by 2nd PMT.