Name: ______________________   Student Number: _____ - _____ - ________.

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1. (10 points) When a camera zooms in, it can be better modeled using an orthographic model. Please explain the reason. Please describe the data collection procedure using a perspective camera with zoom control so that the Tomasi-Kanade factorization algorithm is applicable.

It can be observed that when the camera zooms in, the focal length increases. When the focal length is increased to infinity, it becomes an orthographic projection. The larger is the focal length, the less the scene depth affects the projection.

Tomasi-Kanade algorithm works for orthographic projection. We can zoom in the camera and take pictures of the scene and then apply the algorithm.
2. (10 points) Suppose in the world coordinate system, the first camera center is (2,2,2) and the associated three axes are \((\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0),\) and \((0,0,1)\). It is also known that in the second camera coordinate system, the world coordinate center is at \((-1,0,0)\) and its associated axes are \((0,1,0), (-1,0,0),\) and \((0,0,1)\). Please describe how to represent the second camera coordinate system in the first camera coordinate system (center and axes)?

If we use the notion \(P'=R(P-T)\), then

World to camera 1: \(P_1 = R_{w\to 1}(P_w - T_{w\to 1})\),

where \(R_{w\to 1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\) and \(T_{w\to 1} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\)

Camera 2 to world: \(P_w = R_{2\to w}(P_2 - T_{2\to w})\)

where \(R_{2\to w} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\) and \(T_{2\to w} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\)

Camera 2 to camera 1:

\(P_1 = R_{w\to 1}(P_2 - T_{2\to w}) - T_{w\to 1}\)

Camera 1 to camera 2

\(P_2 = R_{2\to w}^t(R_{w\to 1}^tP_1 + T_{w\to 1}) + T_{2\to w}\)

Therefore \(R_{1\to 2} = R_{2\to w}^tR_{w\to 1}^t = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\)

\(T_{1\to 2} = -R_{w\to 1}T_{w\to 1} - R_{w\to 1}R_{2\to w}T_{2\to w}\)

Therefore, the center of the second camera coordinate system in the first camera coordinate system is \([-5\sqrt{2}/2, -\sqrt{2}/2, -2]^t\) and the three axes are \([-\sqrt{2}/2, -\sqrt{2}/2, 0]^t, [\sqrt{2}/2, -\sqrt{2}/2, 0]^t,\) and \([0, 0, 1]^t\).
3. (10 points) Please design an algorithm for detecting pairs of parallel lines in an edge image.

Three parameters are needed for describing a pair of lines. The angle $\theta$ of the normal vector, the distance for the two lines $d_1$ and $d_2$.

Using the generalized Hough transform, we discretize the 3D parameter space into cells. For each edge point $(x, y)$, we need to find all the $(\theta, d_1, d_2)$ that pass through it.

Since we know that for a line passing through $(x, y)$, it must satisfy $d = l \cos(\theta - \theta_0)$, where $l$ is the distance from $(x, y)$ to the origin point and $\theta_0$ is the angle between $(x, y)$ and the x axis, therefore, for $(x, y)$, we increase the counter in cells that satisfy $d_1 = l \cos(\theta - \theta_0)$ or $d_2 = l \cos(\theta - \theta_0)$. 
4. (10 points) Imagine in a 2D world, a scene point $P$ can be described using its 2D coordinates $(x, y)$. An orthographic camera in this 2D world has a single row of pixels. If the 2D transformation from the world coordinate system to the camera coordinate system is $P_c = R(P - T)$, where $R$ is a 2x2 rotation matrix in the form of $R = \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix}$, $T$ is a 2D translation vector, then the projection matrix from the 2D world coordinates to the homogeneous image coordinates is $M = \begin{bmatrix} s & u_0 \\ 0 & 1 \end{bmatrix}$, where $u_0$ is the principal point, $s$ is the scale factor.

Suppose there are two cameras at $R_1, T_1$ and $R_2, T_2$, and all the scene points lie on a line $n^T P = d$ in the world coordinate system. Please derive the transformation between the two 1D images.

![Figure 2. The 2D world and the camera-line system.](image)

This problem is different from the problem in the last year’s midterm. To solve this problem, the basic idea is to find the 2D point on the line from $p^{(1)}$ and then project it into the second camera.

It is obvious that $p^{(1)} = s r_1^{(1)T} (P - T_1) + u_0 \iff s r_1^{(1)T} (I - T_1 n_1^T/d)P = p^{(1)} - u_0$ given that $n^T P = d$.

Together we have $\begin{bmatrix} n^T \\ s r_1^{(1)T} (I - T_1 n_1^T/d) \end{bmatrix} P = \begin{bmatrix} d \\ p^{(1)} - u_0 \end{bmatrix}$, therefore,

$P = \begin{bmatrix} n^T \\ s r_1^{(1)T} (I - T_1 n_1^T/d) \end{bmatrix}^{-1} \begin{bmatrix} d \\ p^{(1)} - u_0 \end{bmatrix}$

Project $P$ into the second camera.
\[ p^{(2)} = sr_1^{(2)T}(P - T_2) + u_0 \]

\[
= sr_1^{(2)T}\left[n^T \left( \frac{sr_1^{(1)T}(I - T_1n^T)}{d} \right) \right]^{-1} \left[p^{(1)} - u_0 \right] - T_2 + u_0
\]

\[
= sr_1^{(2)T}\left[n^T \left( \frac{sr_1^{(1)T}(I - T_1n^T)}{d} \right) \right]^{-1} \left[p^{(1)} - u_0 \right] - sr_1^{(2)T}T_2 + u_0
\]

This can be written as \( p^{(2)} = ap^{(1)} + b \).
5. (10 points) Suppose the optical flow is caused by a frontal rotating plane (the normal vector of the plane is parallel to the optical axis). Please derive the direct method (based on the optical flow equation) for estimating the center and the angle of the 2D rotation.

We want to recover the center and the angle of the 2D rotation, which are denoted as \((x_0, y_0)\) and \(\omega_z\). The key idea to solve this problem is to represent the optical flow in terms of \((x_0, y_0)\) and \(\omega_z\). This can be done as

\[
[u, v]^T = \omega_z[-y + y_0, x - x_0]^T = \omega_z[-y, x]^T + \omega_z[y_0, -x_0]^T.
\]

This becomes linear if we denote \([T_x, T_y]^T = [\omega_z, -\omega_z x_0]^T\). As the result

\[
\begin{bmatrix}
u
\end{bmatrix} = -y \omega_z + \begin{bmatrix} T_x \\ T_y \end{bmatrix} \begin{bmatrix} -y & 1 \\ x & 0 \end{bmatrix} \begin{bmatrix} \omega_z \\ T_x \\ T_y \end{bmatrix}
\]

The optical flow equation \(I_x I_y \begin{bmatrix} u \\ v \end{bmatrix} = I_t\) can now be written as

\[
\begin{bmatrix} I_x \\ I_y \end{bmatrix} \begin{bmatrix} -y & 1 \\ x & 0 \end{bmatrix} \begin{bmatrix} \omega_z \\ T_x \\ T_y \end{bmatrix} = I_t \quad \text{or} \quad -y I_x + x I_y \begin{bmatrix} I_x \\ I_y \end{bmatrix} \begin{bmatrix} \omega_z \\ T_x \\ T_y \end{bmatrix} = I_t
\]

So this is the constraint for each point. We stack all the point together and solve for \(\omega_z, T_x,\) and \(T_y\). The rotation center is \([x_0, y_0]^T = [-T_y / \omega_z, T_x / \omega_z]^T\).
6. (10 points) What is the Kalman gain $K$ when the system noise $Q$ is 0? What is
the Kalman gain $K$ when the measurement noise $R$ is 0? If $Q$ describes how
good the dynamic model is and $R$ describes how accurate the measurement is,
please explain the effect of $Q$ and $R$ on the Kalman gain.

In the Kalman filter

$$K_k = P_k^{-1}H_k^T(H_kP_k^{-1}H_k^T + R_k)^{-1}$$
$$\hat{x}_k = \hat{x}_k^+ + K_k(z_k - H_k\hat{x}_k^-)$$

When $R$ is 0, then $K_k = P_k^{-1}H_k^T(H_kP_k^{-1}H_k^T)^{-1}$. Therefore $H_kK_k = I$

When $Q$ is 0, then because $P_{k+1} = \Phi_kP_k\Phi_k^T + Q_k$ and $P_k = (I - K_kH_k)P_k^-$, if $P_0 = 0$, then

$P_k = 0$. Since $K_k = P_kH_k^TR_k^{-1}$, therefore $K_k = 0$.

If the system is accurately modeled, then $Q$ is close to 0. We should trust the prediction
more and the gain should be small. If the measurement is accurate, then $R$ is close to 0.
We should trust the measurement more and the gain is close to 1. The gain is controlled
by the ratio between $Q$ and $R$. 