Shape from Shading (2)

CMPE 264: Image Analysis and Computer Vision
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Shape from shading

The fundamental equation of shape from shading is

\[ R_{\rho,i}(x, y) = \frac{\rho}{\sqrt{1 + p^2 + q^2}} i^T [-p, -q, 1]^T \]

where \( p = \partial Z / \partial x \) and \( q = \partial Z / \partial y \)

The problem of shape from shading

- Given an image \( E(x, y) \), and full knowledge of the parameters \( \rho \) and \( i \) relative to the available image, reconstruct the surface slopes \( p \) and \( q \), and the surface \( Z = Z(x, y) \)

![Image](image_url)
Shape from shading

- For each pixel, there are two unknown variables and one known quantity. Shape from shading is a highly underconstrained problem.

Solution

- To make the problem tractable, we add a constraint that a “smoother” surface is preferred. This soft constraint can be translated into a regularization term in an optimization problem.
- Variational method for shape from shading: we will find the smoothest solution that satisfies the fundamental equation. In other words, the solution should minimize the following energy function:

\[ \varepsilon = \int \{(E(x, y) - R(p, q))^2 + \lambda(p_x^2 + p_y^2 + q_x^2 + q_y^2)\}dxdy \]
Euler-Lagrange equation

- Calculus of variations
  - A branch of mathematics which is a sort of generalization of calculus. It seeks to find the path, curve, surface, etc. for which a given function has a stationary value (where the derivative vanishes). In physical problems, the stationary value is usually a minimum or a maximum.

- If $J$ is defined as
  \[ J = \int f(t, q, \dot{q}) dt \]

  where $\dot{q} = \frac{dq}{dt}$. The derivative of $J$ vanishes if the Euler-Lagrange equation
  \[ \frac{\partial f}{\partial q} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{q}} \right) = 0 \]

  is satisfied. If the time-derivative $\dot{q}$ is replaced by space-derivative $q_x$, the equation becomes
  \[ \frac{\partial f}{\partial q} - \frac{d}{dx} \left( \frac{\partial f}{\partial q_x} \right) = 0 \]
Euler-Lagrange equation

In our case, the equation is

\[ \varepsilon = \int \left\{ (E(x, y) - R(p, q))^2 + \lambda (p_x^2 + p_y^2 + q_x^2 + q_y^2) \right\} dxdy \]

Using the Euler-Lagrange equation with two independent variables, the stationary value can be achieved when

\[ \frac{\partial f}{\partial p} - \frac{d}{dx} \left( \frac{\partial f}{\partial p_x} \right) - \frac{d}{dy} \left( \frac{\partial f}{\partial p_y} \right) = 0 \quad \text{and} \quad \frac{\partial f}{\partial q} - \frac{d}{dx} \left( \frac{\partial f}{\partial q_x} \right) - \frac{d}{dy} \left( \frac{\partial f}{\partial q_y} \right) = 0 \]

where

\[ f(x, y, p, q, p_x, p_y, q_x, q_y) = (E(x, y) - R(p, q))^2 + \lambda (p_x^2 + p_y^2 + q_x^2 + q_y^2) \]

Therefore,

\[ -2(E - R) \frac{\partial R}{\partial p} - 2\lambda p_{xx} - 2\lambda p_{yy} = 0 \quad \text{and} \quad -2(E - R) \frac{\partial R}{\partial q} - 2\lambda q_{xx} - 2\lambda q_{yy} = 0 \]
The discrete form

The equations

\[-2(E - R) \frac{\partial R}{\partial p} - 2\lambda p_{xx} - 2\lambda p_{yy} = 0 \quad \text{and} \quad -2(E - R) \frac{\partial R}{\partial q} - 2\lambda q_{xx} - 2\lambda q_{yy} = 0\]

can be simplified as

\[p_{xx} + p_{yy} = -\frac{1}{\lambda} (E - R) \frac{\partial R}{\partial p} \quad \text{and} \quad q_{xx} + q_{yy} = -\frac{1}{\lambda} (E - R) \frac{\partial R}{\partial q}\]

We need to find function \(p(x,y)\) and \(q(x,y)\) to satisfy these equations.

The discrete version of these equations are

\[-4p_{i,j} + p_{i,j+1} + p_{i,j-1} + p_{i+1,j} + p_{i-1,j} = -\frac{1}{\lambda} (E(i,j) - R(p_{i,j},q_{i,j})) \frac{\partial R}{\partial p} \bigg|_{p_{i,j},q_{i,j}}\]

\[-4q_{i,j} + q_{i,j+1} + q_{i,j-1} + q_{i+1,j} + q_{i-1,j} = -\frac{1}{\lambda} (E(i,j) - R(p_{i,j},q_{i,j})) \frac{\partial R}{\partial q} \bigg|_{p_{i,j},q_{i,j}}\]

We need to find \(q_{i,j}\) and \(p_{i,j}\) that satisfy these equations and find the surface \(Z(x,y)\).
An iterative algorithm

The previous discrete equations can be rewritten as

\[ p_{i,j} = \bar{p}_{i,j} + \frac{1}{4\lambda} (E(i, j) - R(p_{i,j}, q_{i,j})) \frac{\partial R}{\partial p} \bigg|_{p_{i,j}, q_{i,j}} \]

\[ q_{i,j} = \bar{q}_{i,j} + \frac{1}{4\lambda} (E(i, j) - R(p_{i,j}, q_{i,j})) \frac{\partial R}{\partial q} \bigg|_{p_{i,j}, q_{i,j}} \]

where

\[ \bar{p}_{i,j} = \frac{p_{i,j+1} + p_{i,j-1} + p_{i+1,j} + p_{i-1,j}}{4} \]

\[ \bar{q}_{i,j} = \frac{q_{i,j+1} + q_{i,j-1} + q_{i+1,j} + q_{i-1,j}}{4} \]

\( q_{i,j} \) and \( p_{i,j} \) can be solved by starting from some initial solution at step 0, and advancing from step \( k \) to step \( k+1 \) using the updating rule

\[ p_{i,j}^{k+1} = \bar{p}_{i,j}^k + \frac{1}{4\lambda} (E(i, j) - R(p_{i,j}, q_{i,j})) \frac{\partial R}{\partial p} \bigg|_{p_{i,j}, q_{i,j}}^k \]

\[ q_{i,j}^{k+1} = \bar{q}_{i,j}^k + \frac{1}{4\lambda} (E(i, j) - R(p_{i,j}, q_{i,j})) \frac{\partial R}{\partial q} \bigg|_{p_{i,j}, q_{i,j}}^k \]
Enforcing integrability

Since \( q_{i,j} \) and \( p_{i,j} \) are solved independently, there may not exist a surface \( Z \) so that \( Z_x = p \) and \( Z_x = q \).

- Example: Can you find \( Z_{i,j} \) so that \( p_{i,j} = 0 \) and \( q_{i,j} = (j-100) \) ?

Solution

- After each iteration, we find a revised solution \( p' \) and \( q' \), so that they are integrable and are closest to \( p \) and \( q \).
- Suppose the IFFT of \( p \) and \( q \) are

\[
p = \sum c_p(\omega_x, \omega_y) e^{i(\omega_x x + \omega_y y)}
\]

\[
q = \sum c_q(\omega_x, \omega_y) e^{i(\omega_x x + \omega_y y)}
\]

We can construct the surface function as

\[
Z = \sum c(\omega_x, \omega_y) e^{i(\omega_x x + \omega_y y)}
\]

where

\[
c(\omega_x, \omega_y) = \frac{-i \omega_x c_p(\omega_x, \omega_y) - i \omega_y c_q(\omega_x, \omega_y)}{\omega_x^2 + \omega_y^2}
\]
Enforcing integrability

Then the integrable solution that are closest to \( p \) and \( q \) are

\[
p' = \frac{\partial Z}{\partial x} = \sum (i \omega_x c(\omega_x, \omega_y)) e^{i(\omega_x x + \omega_y y)}
\]

\[
q' = \frac{\partial Z}{\partial y} = \sum (i \omega_y c(\omega_x, \omega_y)) e^{i(\omega_x x + \omega_y y)}
\]

It can be verified by starting from \( p' \) and \( q' \) and using the above equation to obtain the same \( p' \) and \( q' \)
Shape from shading - algorithm

**Shape_from_shading**

- Given the effective albedo, the illuminant direction, and the image, initialize the surface slopes $p$ and $q$ to 0.
- Until a suitable criterion is met, iterate the following two steps
  - Update $p$ and $q$ using (1)
  - Compute the FFT of the updated $p$ and $q$, estimate $Z$ and $p'$ and $q'$ using (2)
  - Output $Z$, $p$, $q$
Figure 9.4 Reconstructions of the surface in Figure 9.2 after 100 (a), 1000 (b) and 2000 (c) iterations. The initial surface was a plane of constant height. The asymmetry of the first two reconstruction is due to the illuminant direction.
Homework

- Derive an algorithm using the variational method to compute optical flow based on the optical flow equation, where the situation is similar to the shape from shading problem. For each pixel, there are two unknown variables and one known quantity.
Derivation of the Euler-Lagrange equation (1)

Since

\[ J = \int f(t, q, \dot{q}) dt \]

therefore

\[ \delta J = \delta \int f(t, q, \dot{q}) dt = \int \left( \frac{\partial f}{\partial q} \delta q + \frac{\partial f}{\partial \dot{q}} \delta \dot{q} \right) dt \]

\[ = \int \left( \frac{\partial f}{\partial q} \delta q + \frac{\partial f}{\partial \dot{q}} \frac{d(\delta q)}{dt} \right) dt \]

(3)

Using the rule of integration by parts \[ \int u dv = uv - \int v du \] yields

\[ \int \frac{\partial f}{\partial \dot{q}} \frac{d(\delta q)}{dt} dt = \int \frac{\partial f}{\partial q} d(\delta q) = \left. \frac{\partial f}{\partial \dot{q}} \frac{d(\delta q)}{dt} \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left( \frac{d}{dt} \frac{\partial f}{\partial \dot{q}} \right) \delta q \]

Since we are varying the path \( q \), not the end points, therefore

\[ \delta q(t_1) = \delta q(t_2) = 0 \] . As the result

\[ \int \frac{\partial f}{\partial \dot{q}} \frac{d(\delta q)}{dt} dt = -\int_{t_1}^{t_2} \left( \frac{d}{dt} \frac{\partial f}{\partial \dot{q}} \right) \delta q \]

(4)
Derivation of the Euler-Lagrange equation (2)

Combining (3) and (4) yields

$$\delta J = \int_{t_1}^{t_2} \left( \frac{\partial f}{\partial q} - \frac{d}{dt} \frac{\partial f}{\partial \dot{q}} \right) \delta q \, dt$$

For a stationary point, for any small $\delta q$, $\delta J = 0$, therefore

$$\frac{\partial f}{\partial q} - \frac{d}{dt} \frac{\partial f}{\partial \dot{q}} = 0$$