Shape from Shading

CMPE 264: Image Analysis and Computer Vision
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Shape from X

Many cues can be used for inferring object shapes from images. Based on the number of images used, there are two categories of methods: methods using multiple images and methods using a single image. A list of so called shape-from-X algorithms are shown in the following table.

<table>
<thead>
<tr>
<th>Shape from</th>
<th>How many images</th>
<th>Method type</th>
<th>Find more in</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stereo</td>
<td>2 or more</td>
<td>passive</td>
<td>Chapter 7</td>
</tr>
<tr>
<td>Motion</td>
<td>a sequence</td>
<td>active/passive</td>
<td>Chapter 8</td>
</tr>
<tr>
<td>Focus/defocus</td>
<td>2 or more</td>
<td>active</td>
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<tr>
<td>Zoom</td>
<td>2 or more</td>
<td>active</td>
<td>Further Readings</td>
</tr>
<tr>
<td>Contours</td>
<td>single</td>
<td>passive</td>
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<td>Texture</td>
<td>single</td>
<td>passive</td>
<td>this chapter</td>
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<tr>
<td>Shading</td>
<td>single</td>
<td>passive</td>
<td>this chapter</td>
</tr>
</tbody>
</table>

Table 9.1 Shape-from-X methods and their classification.
Shape from texture

Figure 9.5  (a) Image of a plane covered by a deterministic texture. (b) The same texture on a curved surface.
Shape from shading

- Shape can be recovered from a single image by human visual system based on the shading information.
- To estimate the shape from a single image, some assumptions have to be made, e.g. constant albedo (surface color).
- This technique is very useful for reconstructing surfaces of planets from photographs acquired by spacecrafts.

![Image](image_url)
Shape from shading - example

(a)

(b)
Reflectance of a Lambertian surface

- The radiance at a 3D point is proportional to the cosine of the angle between the surface normal and the direction of the illuminant

\[ L(P) = R_{\rho,i} = \rho i^T n \]

- \( \rho \) is called the effective albedo: the real albedo times the intensity of illuminant
The pixel intensity at \( \mathbf{p} = [x, y]^T \) is

\[
E(\mathbf{p}) = L(\mathbf{P}) \left( \frac{d}{f} \right)^2 \cos^4 \alpha
\]

If we neglect the constant term and if we assume the optical system has been calibrated to compensate the \( \cos^4 \alpha \) effect, and in addition, if we assume that all the visible points of the surface receive direct illumination, we have the fundamental equation for shape from shading

\[
E(\mathbf{P}) = R_{\rho, i}(\mathbf{n})
\]

Notice that the image intensity is determined only by the surface normal vector.
Representing normal vectors

- If we assume that the scene is far away from the camera, we can use a weak-perspective camera model to describe the projection.
- If we assume the average depth of the scene is \( Z_0 \). The weak-perspective projection can be written as
  
  \[
  x = f \frac{X}{Z_0} \\
  y = f \frac{Y}{Z_0}
  \]

  Therefore, with some rescaling, the scene surface can be considered as a function of the \((x,y)\), or

  \[
  Z = Z(X,Y) = Z(x \frac{Z_0}{f}, y \frac{Z_0}{f}) = Z(x,y)
  \]

  Then the slopes along the x axis and y axis are

  \[
  [1,0, \frac{\partial Z}{\partial X}]^T = [1,0, \frac{\partial Z}{\partial x} \frac{\partial X}{\partial X}]^T = [1,0, \frac{\partial Z}{\partial x} \frac{f}{Z_0}]^T \quad \text{and} \quad [0,1, \frac{\partial Z}{\partial Y} \frac{f}{Z_0}]^T
  \]
Representing normal vectors

- We denote the above vectors as $[1,0,p]^T$ and $[0,1,q]^T$. The normal vector is the cross product of these two vectors, therefore

$$
n = [1,0,p]^T \times [0,1,q]^T = \frac{1}{\sqrt{1+p^2+q^2}}[-p,-q,1]^T
$$

Now we can rewrite the fundamental equation as

$$
E(x,y) = R_{\rho,i}(p,q) = \frac{\rho}{\sqrt{1+p^2+q^2}}i^T[-p,-q,1]^T
$$
The problem of shape from shading

Assumptions

- The imaging system is calibrated so that the $\cos^4 \alpha$ effect is compensated
- All the visible surface receive direct illumination
- The surface is imaged under weak-perspective projection
- The surface can be parameterized as $Z = Z(x, y)$

Shape from shading

- Given the reflectance map of the surface $R_{\rho,i}(p, q)$, and full knowledge of the parameters $\rho$ and $i$ relative to the available image, reconstruct the surface slopes, $p$ and $q$, for which

$$E(x, y) = R_{\rho,i}(p, q)$$

and the surface $Z = Z(x, y)$ such that $p = \frac{\partial Z}{\partial x} \cdot \frac{Z_0}{f}$ and $q = \frac{\partial Z}{\partial y} \cdot \frac{Z_0}{f}$

To simplify the formulation, we will reconstruct a scaled version of the original surface so that

$$p = \frac{\partial Z}{\partial x} \quad \text{and} \quad q = \frac{\partial Z}{\partial y}$$
Finding albedo and illuminant direction

- **Assumption**
  - Albedo is the same for all surface points
  - The surface is Lambertian
  - The direction of the surface normal vectors are uniformly distributed in the 3D space

A normal vector is represented by two angles, $\alpha$ and $\beta$, where $\alpha \in [0, 2\pi]$ and $\beta \in [0, \pi/2]$. As the result, the normal vector is

$$\mathbf{n} = [\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta]^T$$

As seen from the image plane, the probability of a normal vector with $[\alpha, \beta]$ is

$$\frac{\cos \beta}{2\pi}$$
Finding albedo and illuminant direction

- If we also represent the illuminant direction using two angles $\tau$ and $\sigma$, where $\tau \in [0, 2\pi]$ and $\sigma \in [0, \pi/2]$. As the result, the illuminant vector is

$$i = [\cos \tau \sin \sigma, \sin \tau \sin \sigma, \cos \sigma]^T$$

- The image brightness of a surface point with normal angles $\alpha$ and $\beta$ is

$$E(\alpha, \beta) = \rho \mathbf{n} = \rho (\cos \alpha \sin \beta \cos \tau \sin \sigma + \sin \alpha \sin \beta \sin \tau \sin \sigma + \cos \beta \cos \sigma)$$

- The average image intensity becomes

$$\langle E \rangle = \int_0^{\pi/2} \int_0^{2\pi} E(\alpha, \beta) p(\alpha, \beta) d\alpha d\beta$$

$$= \rho \int_0^{\pi/2} \int_0^{2\pi} \left\{ \cos \alpha \sin \beta \cos \tau \sin \sigma + \sin \alpha \sin \beta \sin \tau \sin \sigma + \cos \beta \cos \sigma \right\} \frac{\cos \beta}{2\pi} d\alpha d\beta$$

$$= \rho \int_0^{\pi/2} \int_0^{2\pi} \left\{ \cos \beta \cos \sigma \right\} \frac{\cos \beta}{2\pi} d\alpha d\beta = \rho \cos \sigma \int_0^{\pi/2} \cos^2 \beta d\beta = \frac{\pi}{4} \rho \cos \sigma$$
Finding albedo and illuminant direction

- Similar derivation gives us

\[
\langle E^2 \rangle = \int_0^{2\pi} \int_0^{2\pi} E^2(\alpha, \beta) p(\alpha, \beta) d\alpha d\beta = \frac{1}{6} \rho^2 (1 + 3\cos^2 \sigma)
\]

- From these two equations, we can recover the albedo and the angle \( \sigma \). We can verify that

\[
\gamma = \sqrt{6\pi^2 \langle E^2 \rangle - 48 \langle E \rangle^2} = \sqrt{\pi^2 \rho^2 (1 + 3\cos^2 \sigma) - 3\pi^2 \rho^2 \cos^2 \sigma} = \pi \rho
\]

Therefore

\[
\rho = \frac{\gamma}{\pi}
\]

\[
\cos \sigma = \frac{4 \langle E \rangle}{\gamma}
\]
Finding albedo and illuminant direction

To estimate the angle $\tau$, we compute the average image spatial gradient and estimate $\tau$ as

$$\tan \tau = \frac{\langle \hat{E}_y \rangle}{\langle \hat{E}_x \rangle}$$

Algorithm_Approximate_Albedo_Illuminant

- Compute the average of the image intensity $\langle E \rangle$ and of its square, $\langle E^2 \rangle$
- Compute the average spatial image gradient $\langle \hat{E}_x, \hat{E}_y \rangle$
- Estimate the albedo and the illuminant angle as

$$\rho = \frac{\gamma}{\pi} \quad \cos \sigma = \frac{4 \langle E \rangle}{\gamma} \quad \tan \tau = \frac{\langle \hat{E}_y \rangle}{\langle \hat{E}_x \rangle}$$

where

$$\gamma = \sqrt{6\pi^2 \langle E^2 \rangle - 48 \langle E \rangle^2}$$