Linear Functions and Examples

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Questions?

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad f(x) = \Delta x$$

$$x = a \hat{e}_1 + b \hat{e}_2$$

$$y = c \hat{e}_1 + d \hat{e}_2$$

$$f([0]) = \begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix}$$
Graviometer Prospecting

\[ x_j = P_j - P_{\text{avg}} \]

\[ y_i = \text{measured} \]

\[ \text{gravity} \]

\[ S_i - S_{\text{avg}} \]

\[ x_j < 0 \] 
\[ \text{salt water} \]

\[ x_j > 0 \] 
\[ \text{sedimentary/mineral deposit} \]

\[ x_j < 0 \] 
\[ \text{gas} \]

\[ 10^{-7} \] 
\[ \text{g's} \]
Thermal System

\[ y_t - \text{sensor } t \]

\[ y_i \text{ change in steady state } \]

\[ x_t \text{ heat element } t \]

\[ y_t = \chi x_t \]

Diffusion Process / Reaction Eq. 1
Illumination with Multiple Lamps

$n$ - lamps
$m$ - small patches or shadows

$x_j$ - power to $j$th lamp
$y_i$ - illumination on patch

$$a_{ij} = r_{ij} \max (\cos \theta_{ij}, 0)$$
Signal and Interference Power in Wireless System

- Transmit/receive pairs

- $P_j$ - power of the $j$th transmitter

- $s_i$ - received signal power of $i$th receiver

- $G_{ij}$ path gain from $j$ to $i$

- $S = AP$ if $i=j$ and $0$ otherwise

- $S_i / S_i = SNR$

- $t = Bp$ if $i=j$, $G_{ij}$ otherwise
Cost of Production (1.2)

\[ y = Ax \]

- \( x_j \) price per unit of production input \( j \)
- \( y_i \) cost per unit of product \( i \)
- \( A_{ij} \) unit of production input required to manufacture one unit of product \( i \)
- \( q_i \) quantity of the product
- \( r_j \) total quantity of production input \( j \) that is needed
Cost of Production (2.2)

\[ r = \Lambda^T q \]

\[ r^T x = (\Lambda^T q)^T x = q^T \Lambda^T x = q^T y \]

\[ y = \lambda x \]
Network Traffic and Flows (1.2)

- Flows \( n \) with nodes \( f_1 \ldots f_n \)

- Traffic on link \( i \)

- Flows that pass through link \( i \)

Flow link incidence matrix

\[ A_{ij} = \begin{cases} 1 & \text{Flows go through link } i \\ 0 & \text{otherwise} \end{cases} \]

CMPE 240 – Intro. to Linear Dynamical Systems
Network Traffic and Flows (2.2)

\[ t = A f \]

Link delays

\[ t = A^T d \]

Upstream in the network

\[ f^T x^T d = (x f^T d) = t^T d \]

\( A^T \) has a very relevant physical interpretation
Linearization

\[ f : \mathbb{R}^n \to \mathbb{R}^m \]

If \( f \) is differentiable at \( x_0 \in \mathbb{R}^n \), then:

If \( x \) near \( x_0 \) \( \implies f(x) \) near \( f(x_0) = Df(x_0)(x-x_0) \)

\[ Df(x_0)_{ij} = \frac{df}{dx_j} \bigg|_{x=x_0} \]

\[ y = f(x) \] \( \quad \) \( n \times m \)-linear

\[ y_0 = f(x_0) \]

\[ \delta x = x - x_0 \]

\[ \delta y = y - y_0 \]

\[ \frac{\delta y}{\delta x} = Df(x_0) \]

\( \downarrow \) \( \text{UNSM} \)
Navigation by Ranging (1.2)
Navigation by Ranging (2.2)

\( p \in \mathbb{R}^q \) is a non-linear function of \( (x, y) \in \mathbb{R}^2 \)

\[ p_i (x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2} \]

\[ \mathbf{s}_p = \Lambda \begin{bmatrix} \delta x \\delta y \end{bmatrix} \]

\[ \mathbf{a}_{i2} = \frac{x_0 - p_i}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}} \]

\[ \mathbf{a}_{i2} = \frac{y_0 - q_i}{\text{tan} \theta} \]
Broad Categories of Applications

- Linear model or function $y = Ax$
- Some broad categories of applications:
  - Estimation or inversion
  - Control or design
  - Mapping or transformation

(this list is not exclusive; can have combinations)
Estimation or Inversion

$y_i$ is the $i$th measured or sensor reading (which we know).

$x_j$ is a parameter to be estimated/determined.

$\xi_{ij}$ sensitivity of $j$th parameter to $i$th sensor.

1. Find $x$, given $y$.
2. Find all $x$'s that are consistent with $y$.
3. If no such $x$ exists, check that $y = b_x$, find the least $x$ which is "most consistent" with $y$. 
   
   $y = b_x + D$ non
Control or Design

$x$ is a vector of design parameters or control inputs we can choose.

$y$ is the output or result

Find $x$ such that $y \rightarrow y_{des}$

Find all $x$'s that produce $y \rightarrow y_{des}$

Among all such $x$'s that give $y \rightarrow y_{des}$, choose the best one.
Mapping or Transformation

\[ x \text{ is mapped or transformed by } A \text{ into } y \]

defines if there is an \( x \rightarrow y \)

Said on \( x \) that maps into \( y \)

\( \text{find all } x \)'s that map into \( y \).

decode or undo transformation.
Matrix Multiplication as Mixture of Columns

\[ y = Ax \quad A \in \mathbb{R}^{m \times n} \]

\[ A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \]

\[ x_j \in \mathbb{R}^m \]

\[ y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \]

Each \( x_j \) is a scalar.

\[ A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \quad y = hx = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \]
Unit Vectors

\[ \mathbf{x} = e_j \quad \text{where} \quad j \text{th unit vector} \]

\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ A e_j = a_j \quad \text{\( j \text{th column of } A \)} \]

Inputs \rightarrow \boxed{\text{}} \rightarrow \text{Outputs}
Matrix multiplication as inner product with rows
Geometric Interpretation

\[ y_i = \langle \tilde{a}_i, x \rangle = 0 \]
\[ y_i = \langle \tilde{a}_i, x \rangle = 1 \]
\[ y_i = \langle \tilde{a}_i, x \rangle = 2 \]
\[ y_i = \langle \tilde{a}_i, x \rangle = 3 \]
Block Diagram Representation
Example: Block Upper Triangular
Matrix Multiplication as Composition
Column and Row Interpretations
Inner Product Interpretation
Matrix Multiplication Interpretation via Paths
Questions?