Problem Set 6

1. Harmonic oscillator. The system \( \dot{x} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x \) is called a harmonic oscillator.
   
   (a) Find the eigenvalues, resolvent, and state transition matrix for the harmonic oscillator. Express \( x(t) \) in terms of \( x(0) \).
   
   (b) Sketch the vector field of the harmonic oscillator.
   
   (c) The state trajectories describe circular orbits, \( i.e., \|x(t)\| \) is constant. Verify this fact using the solution from part (a).
   
   (d) You may remember that circular motion (in a plane) is characterized by the velocity vector being orthogonal to the position vector. Verify that this holds for any trajectory of the harmonic oscillator. Use only the differential equation; do not use the explicit solution you found in part (a).

2. Properties of the matrix exponential.
   
   (a) Show that \( e^{A+B} = e^A e^B \) if \( A \) and \( B \) commute, \( i.e., AB = BA \). The converse is also true, \( i.e., \) if \( e^{A+B} = e^A e^B \) then \( A \) and \( B \) commute. (But it is hard to show.)
   
   (b) Carefully show that \( \frac{d}{dt} e^{At} = Ae^{At} = e^{At} A \).

3. Determinant of matrix exponential.
   
   (a) Suppose the eigenvalues of \( A \in \mathbb{R}^{n \times n} \) are \( \lambda_1, \ldots, \lambda_n \). Show that the eigenvalues of \( e^A \) are \( e^{\lambda_1}, \ldots, e^{\lambda_n} \). You can assume that \( A \) is diagonalizable, although it is true in the general case.
   
   (b) Show that \( \det e^A = e^\text{Tr} A \).

   \textit{Hint:} \( \det X \) is the product of the eigenvalues of \( X \), and \( \text{Tr} Y \) is the sum of the eigenvalues of \( Y \).

4. Linear system with a quadrant detector. In this problem we consider the specific system
   
   \[ \dot{x} = Ax = \begin{bmatrix} 0.5 & 1.4 \\ -0.7 & 0.5 \end{bmatrix} x. \]
We have a detector or sensor that gives us the sign of each component of the state
\( x = [x_1 \ x_2]^T \) each second:
\[
y_1(t) = \text{sgn}(x_1(t)), \quad y_2(t) = \text{sgn}(x_2(t)), \quad t = 0, 1, 2, \ldots
\]
where the function \( \text{sgn} : \mathbb{R} \to \mathbb{R} \) is defined by
\[
\text{sgn}(a) = \begin{cases} 
1 & a > 0 \\
0 & a = 0 \\
-1 & a < 0
\end{cases}
\]

There are several ways to think of these sensor measurements. You can think of
\( y(t) = [y_1(t) \ y_2(t)]^T \) as determining which quadrant the state is in at time \( t \) (thus the
name *quadrant detector*). Or, you can think of \( y(t) \) as a one-bit quantized measurement
of the state at time \( t \).

Finally, the problem. You observe the sensor measurements
\[
y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

Based on these measurements, what values could \( y(2) \) possibly take on?

In terms of the quadrants, the problem can be stated as follows. \( x(0) \) is in quadrant IV,
and \( x(1) \) is also in quadrant IV. The question is: which quadrant(s) can \( x(2) \) possibly
be in?

You do not know the initial state \( x(0) \).

Of course, you must completely justify and explain your answer.

5. *Some basic properties of eigenvalues.* Show that
   
   (a) the eigenvalues of \( A \) and \( A^T \) are the same
   (b) \( A \) is invertible if and only if \( A \) does not have a zero eigenvalue
   (c) if the eigenvalues of \( A \) are \( \lambda_1, \ldots, \lambda_n \) and \( A \) is invertible, then the eigenvalues of
       \( A^{-1} \) are \( 1/\lambda_1, \ldots, 1/\lambda_n \),
   (d) the eigenvalues of \( A \) and \( T^{-1}AT \) are the same.

*Hint:* you’ll need to use the facts \( \det AB = \det A \det B \) and \( \det A^{-1} = 1/\det A \) (pro-
vided \( A \) is invertible).

6. *Characteristic polynomial.* Consider the characteristic polynomial \( \chi(s) = \det(sI - A) \)
of the matrix \( A \in \mathbb{R}^{n \times n} \).

   (a) Show that \( \chi \) is *monic*, which means that its leading coefficient is one: \( \chi(s) = s^n + \cdots \).
   (b) Show that the \( s^{n-1} \) coefficient of \( \chi \) is given by \(-\text{Tr} \ A\). (\( \text{Tr} \ X \) is the *trace* of a
       matrix: \( \text{Tr} \ X = \sum_{i=1}^n X_{ii} \)).
(c) Show that the constant coefficient of $X$ is given by $\det(-A)$.
(d) Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $A$, so that

$$X(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n).$$

By equating coefficients show that $a_{n-1} = -\sum_{i=1}^{n} \lambda_i$ and $a_0 = \prod_{i=1}^{n} (-\lambda_i)$.

7. **Left eigenvector properties.** Suppose $w$ is a left eigenvector of $A \in \mathbb{R}^{n\times n}$ with real negative eigenvalue $\lambda$.

(a) Find a simple expression for $w^T e^{At}$.
(b) Let $\alpha < \beta$. The set \{ $z \mid \alpha \leq w^T z \leq \beta$ \} is referred to as a slab. Briefly explain this terminology. Draw a picture in $\mathbb{R}^2$.
(c) Show that the slab \{ $z \mid 0 \leq w^T z \leq \beta$ \} is invariant for $\dot{x} = Ax$.

8. **Some Matlab exercises.** Consider the continuous-time system $\dot{x} = Ax$ with

$$A = \begin{bmatrix}
-0.1005 & 1.0939 & 2.0428 & 4.4599 \\
-1.0880 & -0.1444 & 5.9859 & -3.0481 \\
-2.0510 & -5.9709 & -0.1387 & 1.9229 \\
-4.4575 & 3.0753 & -1.8847 & -0.1164
\end{bmatrix}.$$

(a) What are the eigenvalues of $A$? Is the system stable? You can use the command `eig` in Matlab.
(b) Plot a few trajectories of $x(t)$, i.e., $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$, for a few initial conditions. To do this you can use the matrix exponential command in Matlab `expm` (not `exp` which gives the element-by-element exponential of a matrix), or more directly, the Matlab command `initial` (use `help initial` for details.) Verify that the qualitative behavior of the system is consistent with the eigenvalues you found in part (8a).
(c) Find the matrix $Z$ such that $Zx(t)$ gives $x(t + 15)$. Thus, $Z$ is the ‘15 seconds forward predictor matrix’.
(d) Find the matrix $Y$ such that $Yx(t)$ gives $x(t - 20)$. Thus $Y$ reconstructs what the state was 20 seconds ago.
(e) Briefly comment on the size of the elements of the matrices $Y$ and $Z$.
(f) Find $x(0)$ such that $x(10) = [1 \ 1 \ 1 \ 1]^T$. 