1. **Spectral resolution of the identity.** Suppose $A \in \mathbb{R}^{n \times n}$ has $n$ linearly independent eigenvectors $p_1, \ldots, p_n$, $p_i^T p_i = 1$, $i = 1, \ldots, n$, with associated eigenvalues $\lambda_i$. Let $P = [p_1 \cdots p_n]$ and $Q = P^{-1}$. Let $q_i^T$ be the $i$th row of $Q$.

   (a) Let $R_k = p_k q_k^T$. What is the range of $R_k$? What is the rank of $R_k$? Can you describe the null space of $R_k$?

   (b) Show that $R_i R_j = 0$ for $i \neq j$. What is $R_2^2$?

   (c) Show that
   
   $$\left(sI - A\right)^{-1} = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}.$$ 

   Note that this is a partial fraction expansion of $(sI - A)^{-1}$. For this reason the $R_i$’s are called the *residue* matrices of $A$.

   (d) Show that $R_1 + \cdots + R_n = I$. For this reason the residue matrices are said to constitute a resolution of the identity.

   (e) Find the residue matrices for
   
   $$A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$$ 

   both ways described above (*i.e.*, find $P$ and $Q$ and then calculate the $R$’s, and then do a partial fraction expansion of $(sI - A)^{-1}$ to find the $R$’s).

2. Consider the discrete-time system $x(t+1) = Ax(t)$, where $x(t) \in \mathbb{R}^n$.

   (a) Find $x(t)$ in terms of $x(0)$.

   (b) Suppose that $\det(zI - A) = z^n$. What are the eigenvalues of $A$? What (if anything) can you say about $x(k)$ for $k < n$ and $k \geq n$, without knowing $x(0)$?

3. Consider the linear dynamical system $\dot{x} = Ax$ where $A \in \mathbb{R}^{n \times n}$ is diagonalizable with eigenvalues $\lambda_i$, eigenvectors $v_i$, and left eigenvectors $w_i$ for $i = 1, \ldots, n$. Assume that $\lambda_1 > 0$ and $\Re \lambda_i < 0$ for $i = 2, \ldots, n$. Describe the trajectories qualitatively. Specifically, what happens to $x(t)$ as $t \to \infty$? Give the answer geometrically, in terms of $x(0)$.
4. **Stability of a periodic system.** Consider the linear dynamical system \( \dot{x} = A(t)x \) where
\[
A(t) = \begin{cases}
A_1 & 2n \leq t < 2n + 1, \ n = 0, 1, 2, \ldots \\
A_2 & 2n + 1 \leq t < 2n + 2, \ n = 0, 1, 2, \ldots 
\end{cases}
\]
In other words, \( A(t) \) switches between the two values \( A_1 \) and \( A_2 \) every second. We say that this (time-varying) linear dynamical system is stable if every trajectory converges to zero, i.e., we have \( x(t) \to 0 \) as \( t \to \infty \) for any \( x(0) \).

Find the conditions on \( A_1 \) and \( A_2 \) under which the periodic system is stable. Your conditions should be as explicit as possible.

5. **Rate of a Markov code.** Consider the Markov language described in exercise 2-3, with five symbols 1, 2, 3, 4, 5, and the following symbol transition rules:
   - 1 must be followed by 2 or 3
   - 2 must be followed by 2 or 5
   - 3 must be followed by 1
   - 4 must be followed by 4 or 2 or 5
   - 5 must be followed by 1 or 3

(a) **The rate of the code.** Let \( K_N \) denote the number of allowed sequences of length \( N \). The number
\[
R = \lim_{N \to \infty} \frac{\log_2 K_N}{N}
\]
(if it exists) is called the **rate** of the code, in bits per symbol. Find the rate of this code. Compare it to the rate of the code which consists of all sequences from an alphabet of 5 symbols (i.e., with no restrictions on which symbols can follow which symbols).

(b) **Asymptotic fraction of sequences with a given starting or ending symbol.** Let \( F_{N,i} \) denote the number of allowed sequences of length \( N \) that start with symbol \( i \), and let \( G_{N,i} \) denote the number of allowed sequences of length \( N \) that end with symbol \( i \). Thus, we have
\[
F_{N,1} + \cdots + F_{N,5} = G_{N,1} + \cdots + G_{N,5} = K_N.
\]
Find the asymptotic fractions
\[
f_i = \lim_{N \to \infty} F_{N,i}/K_N, \quad g_i = \lim_{N \to \infty} G_{N,i}/K_N.
\]
We won’t give full credit for answers obtained by simple simulation or relatively mindless computation; we want to see (and understand) your method.

6. **Squareroot and logarithm of a (diagonalizable) matrix.** Suppose that \( A \in \mathbb{R}^{n \times n} \) is diagonalizable. Therefore, an invertible matrix \( T \in \mathbb{C}^{n \times n} \) and diagonal matrix \( \Lambda \in \mathbb{C}^{n \times n} \) exist such that \( A = T\Lambda T^{-1} \). Let \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \).
(a) We say \( B \in \mathbb{R}^{n \times n} \) is a squareroot of \( A \) if \( B^2 = A \). Let \( \mu_i \) satisfy \( \mu_i^2 = \lambda_i \). Show that \( B = T \text{diag}(\mu_1, \ldots, \mu_n)T^{-1} \) is a squareroot of \( A \). A squareroot is sometimes denoted \( A^{1/2} \) (but note that there are in general many squareroots of a matrix).

When \( \lambda_i \) are real and nonnegative, it is conventional to take \( \mu_i = \sqrt{\lambda_i} \) (i.e., the nonnegative squareroot), so in this case \( A^{1/2} \) is unambiguous.

(b) We say \( B \) is a logarithm of \( A \) if \( e^B = A \), and we write \( B = \log A \). Following the idea of part a, find an expression for a logarithm of \( A \) (which you can assume is invertible). Is the logarithm unique? What if we insist on \( B \) being real?

7. Suppose \( \dot{x} = Ax \) with \( A \in \mathbb{R}^{n \times n} \). Two one-second experiments are performed. In the first, \( x(0) = [1 \ 1]^T \) and \( x(1) = [4 \ -2]^T \). In the second, \( x(0) = [1 \ 2]^T \) and \( x(1) = [5 \ -2]^T \).

(a) Find \( x(1) \) and \( x(2) \), given \( x(0) = [3 \ -1]^T \).

(b) Find \( A \), by first computing the matrix exponential.

(c) Either find \( x(1.5) \) or explain why you cannot \( x(0) = [3 \ -1]^T \).

(d) More generally, for \( \dot{x} = Ax \) with \( A \in \mathbb{R}^{n \times n} \), describe a procedure for finding \( A \) using experiments with different initial values. What conditions must be satisfied for your procedure to work?

8. Affine dynamical systems. A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is called affine if it is a linear function plus a constant, i.e., of the form \( f(x) = Ax + b \). Affine functions are more general than linear functions, which result when \( b = 0 \). We can generalize linear dynamical systems to affine dynamical systems, which have the form

\[
\dot{x} = Ax + Bu + f, \quad y = Cx + Du + g.
\]

Fortunately we don’t need a whole new theory for (or course on) affine systems; a simple shift of coordinates converts it to a linear dynamical system. Assuming \( A \) is invertible, define \( \tilde{x} = x + A^{-1}f \) and \( \tilde{y} = y - g + CA^{-1}f \). Show that \( \tilde{x}, u, \) and \( \tilde{y} \) are the state, input, and output of a linear dynamical system.

9. Two separate experiments are performed for \( t \geq 0 \) on the single-input single-output (SISO) linear system

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x(0) = [1 \ 2 \ -1]^T
\]

(the initial condition is the same in each experiment). In the first experiment, \( u(t) = e^{-t} \) and the resulting output is \( y(t) = e^{-3t} + e^{-2t} \). In the second, \( u(t) = e^{-3t} \) and the resulting output is \( y(t) = 3e^{-3t} - e^{-2t} \).

(a) Can you determine the transfer function \( C(sI - A)^{-1}B + D \) from this information? If it is possible, do so. If not, find two linear systems consistent with all the data given which have different transfer functions.

(b) Can you determine \( A, B, C, \) or \( D \)?
10. **A method for rapidly driving the state to zero.** We consider the discrete-time linear dynamical system

\[ x(t + 1) = Ax(t) + Bu(t), \]

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times k}, k < n, \) is full rank.

The goal is to choose an input \( u \) that causes \( x(t) \) to converge to zero as \( t \to \infty \). An engineer proposes the following simple method: at time \( t \), choose \( u(t) \) that minimizes \( \|x(t + 1)\| \). The engineer argues that this scheme will work well, since the norm of the state is made as small as possible at every step. In this problem you will analyze this scheme.

(a) Find an explicit expression for the proposed input \( u(t) \) in terms of \( x(t), A, \) and \( B \).

(b) Now consider the linear dynamical system \( x(t + 1) = Ax(t) + Bu(t) \) with \( u(t) \) given by the proposed scheme (i.e., as found in (10a)). Show that \( x \) satisfies an autonomous linear dynamical system equation \( x(t + 1) = Fx(t) \). Express the matrix \( F \) explicitly in terms of \( A \) and \( B \).

(c) Now consider a specific case:

\[
A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Compare the behavior of \( x(t + 1) = Ax(t) \) (i.e., the orginal system with \( u(t) = 0 \)) and \( x(t + 1) = Fx(t) \) (i.e., the original system with \( u(t) \) chosen by the scheme described above) for a few initial conditions. Determine whether each of these systems is stable.

11. **Analysis of a power control algorithm.** In this problem we consider again the power control method described in homework problem 1-1. Please refer to this problem for the setup and background. In that problem, you expressed the power control method as a discrete-time linear dynamical system, and simulated it for a specific set of parameters, with several values of initial power levels, and two target SINRs. You found that for the target SINR value \( \gamma = 3 \), the powers converged to values for which each SINR exceeded \( \gamma \), no matter what the initial power was, whereas for the larger target SINR value \( \gamma = 5 \), the powers appeared to diverge, and the SINRs did not appear to converge.

You are going to analyze this, now that you know alot more about linear systems.

(a) **Explain the simulations.** Explain your simulation results from the problem 1(b) for the given values of \( G, \alpha, \sigma, \) and the two SINR threshold levels \( \gamma = 3 \) and \( \gamma = 5 \).

(b) **Critical SINR threshold level.** Let us consider fixed values of \( G, \alpha, \) and \( \sigma \). It turns out that the power control algorithm works provided the SINR threshold \( \gamma \) is less than some critical value \( \gamma_{\text{crit}} \) (which might depend on \( G, \alpha, \sigma \)), and doesn’t work for \( \gamma > \gamma_{\text{crit}} \). (‘Works’ means that no matter what the initial powers are, they converge to values for which each SINR exceeds \( \gamma \).)
Find an expression for $\gamma_{\text{crit}}$ in terms of $G \in \mathbb{R}^{n \times n}$, $\alpha$, and $\sigma$. Give the simplest expression you can. Of course you must explain how you came up with your expression.