Menger’s Theorems

Given \( u, v \in V(G), u \neq v \).

A subset \( S \subseteq V(G) - \{u, v\} \) is a \( u-v \) separating set
if \( u \) and \( v \) are not connected in \( G - S \).

\( u-v \) separating sets??

\{a,b\} ?
\{a,d,e\} ?
\{b,d,f\} ?

\begin{center}
\includegraphics[width=0.5\textwidth]{mengers_theorems.png}
\end{center}

Thm 8.3

\( S \) is a \( u-v \) separating set of vertices \( \text{every } u-v \) path
has an internal vertex in \( S \).

\( F \) is a \( u-v \) separating set of edges \( \text{every } u-v \) path
contains an edge in \( F \).

Proof: \( \text{ } \)

If \( S \) is a \( u-v \) separating set then \( u \) and \( v \) are not
connected in \( G-S \).

There is no \( u-v \) path in \( G-S \).

Any \( u-v \) path in \( G \) must have a vertex in \( S \).

\( u \) and \( v \) are not in \( S \), so every \( u-v \) path has
an internal vertex in \( S \).

\begin{center}
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\end{center}

Proof: \( \text{ } \) (continued)

If \( F \) is a \( u-v \) separating set then \( u \) and \( v \) are not
connected in \( G-F \).

There is no \( u-v \) path in \( G-F \).

Any \( u-v \) path in \( G \) must have an edge in \( F \).

If every \( u-v \) path has an internal vertex
(edge) in \( S(F) \), then there is no \( u-v \) path in
\( G-S \) (\( G-F \)).

\( u \) and \( v \) are not connected in \( G-S \) (\( G-F \)).

\( S(F) \) is a \( u-v \) separating set of vertices (edges). \( \square \)
Menger’s Thm 8.4
Assume \( u, v \in V(G) \), \( u \neq v \) and \( (u,v) \in E(G) \).
The maximum number of internally disjoint \( u \cdot v \) paths in \( G \) is equal to the minimum number of vertices in a \( u \cdot v \) separating set.

Corollary (Thm 8.5)
A simple graph is \( n \)-connected every pair of distinct vertices have \( n \) internally disjoint paths.

\( n \)-connected
- either \( G \cong K_{n+1} \) or no cut set of size \( < n \)
- no pair of vertices has a separating set of size \( < n \)
- every pair non-adjacent vertices has at least \( n \) internally disjoint paths.

Proof: (continued)
To show \( p \geq k \), given a graph \( G \) and two non-adjacent vertices, \( u \) and \( v \), construct the network \( N=(D,c(),s,t) \) as follows:

For all vertices \( w \neq u,v \)

\[
+m = 2 \cdot |V|
\]

\[
+m = 2 \cdot |V|
\]
Proof: (continued)

Let $d$ be the value of a maximum flow $f$ for $N$.

Claim: $p \geq d \geq k$.

Follow a unit of flow from $s$ to $t$ in $N$ to obtain a directed walk $W$ of $N$.

No vertex can be repeated since there is only one arc leaving each $W_{in}$ and it has capacity 1.

$W$ defines a $u$ to $v$ path in $G$.

Remove the unit of flow from $f$ for all arcs in $W$ and remove the internal arcs for all vertices in $W$.

Repeat until the value of the flow is 0.

Proof: (continued)

The result is $d$ internally disjoint $u$-$v$ paths in $G$ since each internal arc can be used by only 1 unit of flow allowing each vertex of $V \setminus \{u,v\}$ to appear in only one path.

So $p \geq d$.

Let $X$ be a minimum cut of $N$.

$d = c(X, \overline{X}) \leq |V| - 2$, since removing all internal arcs disconnects $t$ from $s$.

So $A(X, \overline{X})$ contains only arcs of capacity 1.
Proof: (continued)

Let $S = \{ w \mid w_x \in X, \text{ and } w_{out} \in X \}$

$S$ is a $u$-$v$ separating set of size $d$ since any $u$-$v$ path in $G-S$ would correspond to an $s$-$t$ path in $N$ without any arcs in $A(X, X)$.

d = c(X, X) = |S| \geq k.

So $p \geq d \geq k \geq p$. \qed

Example 1 $S = \{a, c\}$, $u$-$v$ separating set of size 2

Menger's Thm 8.5
Assume $u, v \in V(G)$, $u \neq v$.

The maximum number of edge disjoint $u$-$v$ paths in $G$ is equal to the minimum number of edges in a $u$-$v$ separating set.

Proof:
Let $p =$ maximum number of edge disjoint $u$-$v$ paths in $G$.
Let $k =$ the minimum number of edges in a $u$-$v$ separating set.

Then $p \leq k$ since at least one edge from each of the $p$ edge disjoint paths must be removed to disconnected $u$ and $v$. 

Example 2 $S' = \{b, c\}$, $u$-$v$ separating set of size 2
To show $p \geq k$, given a graph $G$ and two distinct vertices, $u$ and $v$, construct the network $N=(D,c(),s,t)$ as follows:

For all vertices $w \neq u,v$

**Proof:** (continued)

**Example**

Let $d$ be the value of a maximum flow $f$ for $N$.

Claim: $p \geq d \geq k$.

For any pair of adjacent vertices, $a,b$ if there is flow from both $a$ to $b$ and $b$ to $a$, remove it till there is flow in one direction only.
Proof: (continued)

Follow a unit of flow from \( s \) to \( t \) in \( N \) to obtain a directed walk \( W \) of \( N \).

No arc can be repeated since each arc has capacity 1.

\( W \) defines a \( u \) to \( v \) trail in \( G \) since no edge can be repeated (anti-parallel arcs cannot occur).

Remove the unit of flow from \( f \) for all arcs in \( W \) and remove the arcs for all edges in \( W \).

Repeat until the value of the flow is 0.

The result is \( d \) edge disjoint \( u-v \) paths in \( G \).

Proof: (continued)

So \( p \geq d \).

Let \( X \) be a minimum cut of \( N \). \( d = c(X, \overline{X}) \)

Let \( F = \{ (x, y) \mid (x, y) \in A(X, \overline{X}) \} \)

\( F \) is a \( u-v \) separating set of size \( d \) since any \( u-v \) path in \( G-F \) would correspond to an \( s-t \) path in \( N \) without any arcs in \( A(X, \overline{X}) \).

\( d = c(X, \overline{X}) = |F| \geq k. \)

So \( p \geq d \geq k \geq p \). \( \square \)

Example

\( S=\{(u,c),(a,b)\} \), \( u-v \) separating set of size 2

Equivalent Theorems for Directed Graphs

Assume \( u,v \in V(G) \), \( u \neq v \) and \( (u,v) \in A(D) \).

The maximum number of internally disjoint \( u-v \) paths in \( D \) is equal to the minimum number of vertices in a \( u-v \) separating set.

Assume \( u,v \in V(G) \), \( u \neq v \).

The maximum number of arc disjoint \( u-v \) paths in \( D \) is equal to the minimum number of arcs in a \( u-v \) separating set.