Euler Tours
A tour of $G$ is a closed walk which includes every edge at least once.

An Euler tour of $G$ is a tour which includes every edge exactly once.

An Euler trail of $G$ is a trail which includes every edge exactly once.

A graph is Eulerian if it has an Euler tour.

Eulerian?
Does it have an Euler trail?

Thm
For a connected graph $G$ the following are equivalent:
1. $G$ is Eulerian.
2. All vertices of $G$ have even degree.
3. The edges of $G$ can be partitioned into cycles.

Thm 3.2 1 if and only if 2
Thm 3.3 1 if and only if 3
Show $1 \iff 2 \iff 3$
**Assume** $G$ is Eulerian.

**Proof:** Let $C$ be an Euler Tour of $G$. Let $v$ be any vertex of $G$.

If $d(v) = 0$, then $v$ has even degree.

If $d(v) > 0$, then $v$ has one or more edges which all appear exactly once in $C$.

Each occurrence of $v$ in $C$ is preceded and followed by an edge of $v$.

Loops can be used both to enter and leave $v$, but non-loops are used either to enter or leave $v$, not both.

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**Proof:** (continued)

$d(v) = 2(\# \text{ loops at } v) + (\# \text{ non-loop edges of } v)$

Since $C$ enters and leaves $v$ the same # of times.

So $d(v)$ is even.

**Proof:** (continued)

By induction on the # of cycles of $G$, $k$.

**Base:** $k = 0$. If $G$ has no cycles, it is a tree.

A tree with at least 2 vertices, must have 2 vertices of degree 1 by Corollary 2.3.

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**Step:** $k > 0$. Let $C$ be a cycle in $G$.

Let $G'$ be the graph $G - E(C)$ (remove the edges of $C$).

The degree of any vertex in $G'$ is either the same as in $G$ or 2 less.

So the 0 edges can be partitioned into 0 cycles.

The degree of any vertex in $G'$ is still even.

$G'$ has less than $k$ cycles.

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**Proof:** (continued)

Each $G_i$ is a connected graph with even degrees having less than $k$ cycles.

By induction the edges of each $G_i$ can be partitioned into cycles.

Let $C_1, C_2, \ldots, C_l$ be the cycles from all of the components of $G'$.

Then $C_1, C_2, \ldots, C_l, C$ is a partition into cycles of $E(G)$.
Proof: 3. 1
Let $C$ and $C'$ be closed trails of $G$ which intersect at $v$, but have no common edges.

$$C = ve_1v_1e_2v_2...v_{k-1}e_kv$$

$$C' = v_1u_1f_1u_2...u_{k-1}f_kv$$

Then $C' = ve_1v_1e_2v_2...v_{k-1}e_kv_1u_1f_1u_2...u_{k-1}f_kv$ is a closed trail with all of the edges of $C$ and $C'$.

Let $C_1C_2...C_m$ be the partition into cycles of $E(G)$.

$C_1C_2...C_m$ is also a partition of $E(G)$ into closed trails.

Since $G$ is connected $C_i$ has a vertex in common with some $C_j$ ($j \neq i$).

Corollary
A connected graph $G$ has an Euler Trail if and only if it has either 2 or 0 vertices of odd degree.

Proof: 2. Let $T$ be the Euler Trail, say from $u$ to $v$. Then $G+e$ (where $e=(u,v)$ is a new edge) is Eulerian ($T+e$ is an Euler Tour) and so has only vertices of even degree.

So either $G$ has 0 vertices of odd degree (if $u=v$) or has exactly 2 vertices of odd degree (if $u \neq v$).

Let $u$ and $v$ be the 2 vertices of odd degree, or if there are none, let $u=v$.

Then $G+e$ where $e=(u,v)$ is a new edge) has only vertices of even degree, so it has an Euler Tour, $C$.

Removing $e$ from $C$ gives a $u-v$ Euler Trail of $G$. □

Finding Euler Tours

$C$: empty cycle at $v_0$
while ($G$ has unmarked edges)
    $v_e$: any vertex on $C$ which has unmarked edges
    $C'$: empty list of edges
    $v_e$: $u$
    do
        $e$: unmarked edge of $v$
        mark $e$
        Append $e$ to $C'$
        $v_e$: other endpoint of $e$
    until ($v=v_0$)
    Splice $C'$ into $C$ at $u$
endwhile
Fleury’s Algorithm
(aka don’t burn your bridges)

ve-v0; Fe-E
Ce-trivial path at v0
while (G has unmarked edges)
  If v has an unmarked edge that is not a bridge of (V,F)
    then ee-unmarked edge of v that is not a bridge of (V,F)
  else ee-any unmarked edge of v
    mark e
    Append e to C
    Fe-F-e
  ve-other endpoint of e
endwhile

How do you tell if an edge of \((V,F)\) is a bridge?

Example of Fleury’s Algorithm

Why does Fleury’s Algorithm work?
At any time \(v\) will have at most one edge which is a bridge of \((V,F)\), (will be used last).
Suppose not: \(e\) and \(f\) are bridges.
Assume \(v_0\) is not in \(G_0\).
All vertices in \(G_1\), currently have even degree.
But then the subgraph \(G_1\), has exactly one vertex, \(v_0\), of odd degree.

Euler Tours in Directed Graphs
A directed tour of \(G\) is a closed directed walk which includes every arc at least once.
A directed Euler tour of \(D\) is a directed tour which includes every arc exactly once.
A directed Euler trail of \(D\) is a directed trail which includes every arc exactly once.
A directed graph is Eulerian if it has a directed Euler tour.
**Thm 7.2**

For a weakly connected directed graph $D$:

$D$ is Eulerian if and only if for all $v$, $id(v) = od(v)$.

$D$ has an open directed Euler Trail from $u$ to $v$ if and only if,

1. $od(u) = id(v) + 1$.
2. $id(v) = od(v) + 1$.
3. for all $w \neq u, v$, $id(w) = od(w)$.

If $D$ is Eulerian then it is strongly connected.

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**Finding Euler Tours in Directed Graphs**

1. Build a DFS tree for the graph.
2. Start traversal at root of the tree.
3. Traverse arcs in reverse direction using the tree arc to the parent last.

The traversal can never get stuck since a path to the root will always exist as long as a vertex has an arc left.

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**Example**
Chinese Postman Problem

Given a weighted graph \( G \), find a minimum cost tour.

Each edge must be traversed.

Equivalent to:

Add a set of edges of minimum weight to \( G \) to make it Eulerian, where the cost of each new edge is the distance between its endpoints in \( G \).

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\begin{align*}
(e,f),(b,d) & \quad \text{cost} 3+3 = 6 \\
(e,b),(d,f) & \quad \text{cost} 2+5 = 7 \\
(e,d),(b,f) & \quad \text{cost} 4+3 = 7
\end{align*}
\]

Solution is an Euler Tour of the modified graph.