Directed Graphs

A directed graph \( D = (V,A) \) consists of
1. a finite non-empty set of vertices \( V = V(D) \)
2. An arc set, \( A = A(G) \), where each arc in \( A \) is assigned an **ordered** pair of vertices, \( a = (u,v) \).

If \( a = (u,v) \) then \( a \) joins \( u \) to \( v \),
\( u \) is the origin, tail, initial vertex of \( a \)
\( v \) is the terminus, head, terminal vertex of \( a \).

### Example \( D = (V,A) \)
\[ V = \{a,b,c,d,e\} \]
\[ A = \{a_1,a_2,a_3,a_4,a_5,a_6\} \]
\[ a_1 = (a,b), \quad a_2 = (b,c), \quad a_3 = (c,d), \quad a_4 = (d,e), \quad a_5 = (e,a) \]

Two arcs \( a = (u,v) \) and \( a' = (x,y) \)
are parallel if \( u = x \) and \( v = y \).
Two arcs \( a = (u,v) \) and \( a' = (x,y) \)
are anti-parallel if \( u = y \) and \( v = x \).

An arc is a **loop** if the initial and terminal vertices are the same.

A directed graph is **simple** if it has no loops nor parallel arcs.

The **underlying undirected graph** of a directed graph is the graph obtained by removing the directions of the arcs.

\[ D = (V,A) \]
\[ G = (V,E) \]

Is the underlying undirected graph of a simple directed graph simple?

### Adjacency Matrix \( A(D) \)
An \( n \times n \) matrix where \( n = \# \text{vertices} \)
\[ a_{ij} = \# \text{arcs from } v_i \text{ to } v_j \]

\[
A(D) = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

No longer always symmetric
For simple graphs, all entries are 0 or 1 and diagonal is 0.
Isomorphisms

\( D_1 \cong D_2 \) are isomorphic

If there exist 1-1 onto functions
\[ f_i : V_i \to V_2 \text{ and } f_i : A_i \to A_2, \]
such that \( \forall a_i \in E_i \)
if \( a_i = (u_i, v_i) \) and \( f_i(a_i) = a_2 \), then \( a_2 = (f_i(u_i), f_i(v_i)) \).

The \textit{indegree} of \( v = id(v) \) is the \# of arcs with \( v \) as the terminal vertex.

The \textit{outdegree} of \( v = od(v) \) is the \# of arcs with \( v \) as the initial vertex.

Handshaking Lemma for Directed Graphs

In any directed graph
\[ \sum_{v \in V} id(v) = \sum_{v \in V} od(v) = |A| \]
Proof: Each arc contributes 1 to each sum.

Directed Paths and Cycles

Directed walks, trails, paths and cycles are defined in the same manner except that the arc directions must be respected.

For example, \( W \) is a directed walk if,
\[ W = v_1 a_1 v_2 a_2 v_3 a_3 \ldots a_n v_n \] where each \( a_i = (v_{i-1}, v_i) \).
\( v_1 \) is the origin of \( W \) \( v_n \) is the terminus (destination) of \( W \)
\( D \) is acyclic if it contains no directed cycles.
Every \( x-y \) directed walk contains as a subsequence an \( x-y \) directed path.

\textbf{Proof}: exactly the same as the undirected version.

Connectivity

\( u \) is \textit{reachable} from \( v \), in \( D \) if there is a \( v-u \) walk (path) in \( D \).
\( u \) and \( v \) are \textit{strongly connected} if both are reachable from the other.

Strong connectivity is an \textit{equivalence} relation on vertices of a directed graph.
1. Reflexive: \( u \) is strongly connected to \( u \) by trivial paths.
2. Symmetric: by definition of strong connectivity
3. Transitive: A \( u-v \) walk can be concatenated with a \( v-z \) walk
to obtain a \( u-z \) walk and a \( z-v \) walk can be concatenated with a \( v-u \) walk to obtain a \( v-u \) walk.

The vertices of a directed graph can be divided into equivalence classes called \textit{strongly connected components}.
Connectivity

$D$ is weakly connected if its underlying undirected graph is connected.
$D$ is unilaterally connected if for every pair of vertices at least one is reachable from the other.
$D$ is strongly connected if every pair of vertices are strongly connected.

Types of Arcs

In a simple graph, $a=(u,v)$ is a

- Tree arc if $p[v]=u$
- Cross arc from $u$ to $v$ have $l[u] < l[v] - 1$
- Back arc if $u$ is a proper ancestor of $v$
- Forward arc if $u$ is a descendant of $v$ but not its child.
BFS Trees

No forward arcs for a BFS tree if the graph is simple.

Cross arcs from $u$ to $v$ have $|d(u)| - |v| - 1$