Chapter 1. An Introduction to Graphs

A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge.

Thus, a graph with $n$ vertices is complete if it has as many edges as possible provided there are no loops and no parallel edges.

If the complete graph has vertices $v_1, \ldots, v_n$ then the edge set can be given by

$$E = \{(v_i, v_j) : v_i \neq v_j; i, j = 1, \ldots, n\}$$

It follows that the graph has $\frac{1}{2}n(n-1)$ edges (since there are $n-1$ edges incident with each of the $n$ vertices $v_i$, so a total of $n \times (n-1)$, but divide by 2 since $(v_i, v_j) = (v_j, v_i)$).

Given any two complete graphs with the same number of vertices, $n$, then they are isomorphic. In fact any pairing off of the vertices gives a corresponding pairing off of the edges and hence an isomorphism. For this reason we speak of the complete graph on $n$ vertices. It is denoted by $K_n$.

Figure 1.13 shows $K_1, \ldots, K_6$.

An empty (or trivial) graph is a graph with no edges.

Let $G$ be a graph. If the vertex set $V$ of $G$ can be partitioned into two nonempty subsets $X$ and $Y$ (i.e., $X \cup Y = V$ and $X \cap Y = \emptyset$) in such a way that each edge of $G$ has one end in $X$ and one end in $Y$ then $G$ is called bipartite. The partition $V = X \cup Y$ is called a bipartition of $G$.

A complete bipartite graph is a simple bipartite graph $G$, with bipartition $V = X \cup Y$, in which every vertex in $X$ is joined to every vertex of $Y$.

If $X$ has $m$ vertices and $Y$ has $n$ vertices, such a graph is denoted by $K_{m,n}$.

Section 1.3. More Definitions

Any complete bipartite graph with a bipartition into two sets of $m$ and $n$ vertices is isomorphic to $K_{m,n}$ — in fact any pairing off of the two sets of $m$ vertices together with any pairing off of the two sets of $n$ vertices will give an isomorphism. In particular, $K_{m,n}$ is (of course) isomorphic to $K_{n,m}$.

Since each of the $m$ vertices in the partition set $X$ of $K_{m,n}$ is adjacent to each of the $n$ vertices in the partition set $Y$, $K_{m,n}$ has $m \times n$ edges.

Note that there is now an unfortunate ambiguity in the use of the word complete, since a complete bipartite graph will not in general be complete. Indeed, as the reader should easily verify, the only complete bipartite graph which is complete is $K_{1,1}$.

Figure 1.14 shows two bipartite graphs. They are not complete bipartite. However, the graphs of Figure 1.15 are complete bipartite.

Exercises for Section 1.3

1.3.1 Make a list of drawings of all the graphs with $n$ vertices and $e$ edges for all $n$, $e$ with $n + e \leq 6$. The list should not include any isomorphic pairs of graphs. Determine which of the graphs on your list are simple. (You should get a total of 65 graphs of which 14 are simple.)
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Figure 1.16: Which of these are isomorphic pairs?

Section 1.4. Vertex Degrees

1.3.2 Determine which of the pairs of graphs in Figure 1.16 are isomorphic pairs. (Give an argument justifying your answer.)

1.3.3 In each collection of three graphs shown in Figure 1.17 there is exactly one isomorphic pair. Find each pair and justify that your answer is correct.

Figure 1.17: Find the odd one out!

1.3.4 Find all nonisomorphic complete bipartite graphs with at most 7 vertices.

1.4 Vertex Degrees

An edge $e$ of a graph $G$ is said to be incident with the vertex $v$ if $v$ is an end vertex of $e$. In this case we also say that $v$ is incident with $e$. Two edges $e$ and $f$ which are incident with a common vertex $v$ are said to be adjacent.
Section 1.5. Subgraphs

1.4.11 Prove that there is no simple graph on four vertices, three of which have degree 3 and the remaining vertex has degree 1.

1.4.12 Let $G$ be a simple regular graph with $n$ vertices and 24 edges. Find all possible values of $n$ and give examples of $G$ in each case.

1.5 Subgraphs

It is often the case that a graph under study is contained within some larger graph also being investigated.

Let $H$ be a graph with vertex set $V(H)$ and edge set $E(H)$ and, similarly, let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Then we say that $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, we also say that $G$ is a supergraph of $H$.

For example, in Figure 1.22, $G_1$ is a subgraph of both $G_2$ and $G_3$ but $G_3$ is not a subgraph of $G_2$.

Any graph isomorphic to a subgraph of $G$ is also referred to as a subgraph of $G$.

If $H$ is a subgraph of $G$ then we write $H \subseteq G$. When $H \subseteq G$ but $H \neq G$, i.e., $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then $H$ is called a proper subgraph of $G$.

A spanning subgraph (or spanning supergraph) of $G$ is a subgraph (or supergraph) $H$ with $V(H) = V(G)$, i.e., $H$ and $G$ have exactly the same vertex set.

It follows easily from the definitions that any simple graph on $n$ vertices is a subgraph of the complete graph $K_n$.

In Figure 1.22, $G_1$ is a proper spanning subgraph of $G_3$.

The simplest types of subgraph of a graph $G$ are those obtained by the deletion of a vertex or an edge and we now define these.
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Section 1.5. Subgraphs

(g) What is the underlying simple graph of $G$? In how many ways can this be obtained?
(h) What is the intersection of the two subgraphs you found in (a) and (b)?
(i) What is the union of the two subgraphs you found in (c) and (d)?

1.5.2 Let $G$ be a simple graph with $n$ vertices. The complement $\overline{G}$ of $G$ is defined to be the simple graph with the same vertex set as $G$ and where two vertices $u$ and $v$ are adjacent precisely when they are not adjacent in $G$. Roughly speaking then, the complement of $G$ can be obtained from the complete graph $K_n$ by “rubbing out” all the edges of $G$. Figure 1.30 shows a graph $G$ on 6 vertices and its complement $\overline{G}$.

![Figure 1.30: A graph and its complement.](image)

Find the complements of the graphs in Figure 1.31.

1.5.3 A simple graph is called self-complementary if it is isomorphic to its own complement.

(a) Find which of the graphs of Figure 1.31 are self-complementary.
(b) Prove that if $G$ is a self-complementary graph with $n$ vertices then $n$ is either $4t$ or $4t + 1$ for some integer $t$. (Hint: consider the number of edges in $K_n$.)

Exercises for Section 1.5

1.5.1 Let $G$ be the graph of Figure 1.29.

(a) Find $G - U$ where $U = \{x_1, x_3, x_5, x_7\}$.
(b) Find $G - F$ where $F = \{e_2, e_4, e_6, e_8, e_{10}, e_{12}\}$.
(c) Find $G[U]$ where $U = \{x_2, x_3, x_4, x_7\}$.
(d) Find $G[F]$ where $F = \{e_1, e_2, e_8, e_{11}\}$.
(e) Find a subgraph $H$ of $G$ isomorphic to $K_3$.
(f) Is there a subgraph of $G$ isomorphic to $K_4$?
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1.5.4 Let $G$ be a simple graph with $n$ vertices and let $\overline{G}$ be its complement.

(a) Prove that, for each vertex $v$ in $G$, $d_G(v) + d_{\overline{G}}(v) = n - 1$.

(b) Suppose that $G$ has exactly one even vertex. How many odd vertices does $G$ have?

1.5.5 Let $G_1$ and $G_2$ be two graphs with no vertex in common. We define the join of $G_1$ and $G_2$, denoted by $G_1 + G_2$, to be the graph with vertex set and edge set given as follows:

\[ V(G_1 + G_2) = V(G_1) \cup V(G_2), \]
\[ E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J \]

where $J = \{ x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2) \}$. Thus $J$ consists of edges which join every vertex of $G_1$ to every vertex of $G_2$. We illustrate this in Figure 1.32.

\[ G_1 + G_2 \]

Figure 1.32: $G_1 + G_2$ is the join of $G_1$ and $G_2$.

(a) Prove that the join of two vertex disjoint complete graphs is a complete graph.

(b) Prove that the complete bipartite graph $K_{m,n}$ is the join of the complements of $K_m$ and $K_n$.

(c) Let $G_1, G_2$ and $G_3$ be three graphs with no vertex common to any pair. Prove that $(G_1 + G_2) + G_3 = G_1 + (G_2 + G_3)$. (This means that the expression $G_1 + G_2 + G_3$ is unambiguous.)

(d) Prove that if $G_1$ and $G_2$ are disjoint simple graphs then the complement of their join is the union of their complements.

Section 1.6. Paths and Cycles

1.6 Paths and Cycles

A walk in a graph $G$ is a finite sequence

\[ W = v_0v_1v_2v_3\ldots v_{k-1}v_k \]

whose terms are alternately vertices and edges such that, for $1 \leq i \leq k$, the edge $e_i$ has ends $v_{i-1}$ and $v_i$. Thus each edge $e_i$ is immediately preceded and succeeded by the two vertices with which it is incident.

We say that the above walk $W$ is a $v_0 - v_k$ walk or a walk from $v_0$ to $v_k$. The vertex $v_0$ is called the origin of the walk $W$, while $v_k$ is called the terminus of $W$. Note that $v_0$ and $v_k$ need not be distinct. The vertices $v_1, \ldots, v_{k-1}$ in the above walk $W$ are called its internal vertices. The integer $k$, the number of edges in the walk, is called the length of $W$.

Note that in a walk there may be repetition of vertices and edges.

In a simple graph, a walk $v_0v_1v_2\ldots v_k$ of its vertices, since for each pair $v_i, v_j$ there is only one possible edge with ends determined by the pair. In fact, even in graphs that are not simple, a walk is often simply denoted by a sequence of vertices

\[ v_0v_1v_2\ldots v_k \]

where consecutive vertices are adjacent. When this is done, it is to be understood that the discussion is valid for every walk with that vertex sequence.

A trivial walk is one containing no edges.

Thus, for any vertex $v$ of $G$, $W = v$ gives a trivial walk. It has length 0.

\[ G \]

Figure 1.33

In Figure 1.33, $W_1 = v_1 e_1 v_2 e_2 v_3 e_6 e_9 e_8 v_5$ and $W_2 = v_1 e_1 v_2 e_7 v_3 e_9 e_8 v_5$ are both walks, of length 5 and 3 respectively, from $v_1$ to $v_3$ and from $v_1$ to $v_2$ respectively.
Section 1.6. Paths and Cycles

(c) Prove that in the Petersen graph of Exercise 1.6.7 above, \( d(u, v) \leq 2 \) for any pair of vertices \( u, v \). (By using the symmetry of the graph you need not look at every pair of vertices.)

1.6.9 Let \( G \) be a connected graph with vertex set \( V \). For each \( v \in V \), the eccentricity of \( v \), denoted by \( e(v) \), is defined by

\[ e(v) = \max \{ d(u, v) : u \in V, u \neq v \}. \]

The radius of \( G \), denoted by \( \text{rad } G \), is defined by

\[ \text{rad } G = \min \{ e(v) : v \in V \}, \]

while the diameter of \( G \), denoted by \( \text{diam } G \), is defined by

\[ \text{diam } G = \max \{ e(v) : v \in V \}. \]

Thus the diameter of \( G \) is given by \( \max \{ d(u, v) : u, v \in V \} \).

(a) Find the radius and the diameter of the graph of Figure 1.39 and the Petersen graph (Figure 1.40).

(b) Prove that for any connected graph \( G \),

\[ \text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G. \]

(c) Find the radius and the diameter of the wheel graphs \( W_n \) of Exercise 1.6.4.

(d) Which simple graphs have diameter 1?

1.6.10 Let \( G \) be a simple connected graph. The square of \( G \), denoted by \( G^2 \), is defined to be the graph with the same vertex set as \( G \) and in which two vertices \( u \) and \( v \) are joined by an edge if and only if in \( G \) we have \( 1 \leq d(u, v) \leq 2 \). An example of a graph \( G \) and its square is shown in Figure 1.41.
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(a) Show that the square of $K_{1,3}$ is $K_4$. Can you find two more graphs whose square is $K_4$?

(b) Draw the squares of the paths $P_4$, $P_5$, $P_6$, the cycles $C_5$, $C_6$ and the wheels $W_5$ and $W_6$.

1.6.11 Let $G$ be a simple graph with $n$ vertices, where $n \geq 2$. Prove that $G$ has two vertices $u$ and $v$ with $d(u) = d(v)$. (Hint: if $G$ is nonempty, consider $G - e$ where $e$ is an edge of $G$, and use induction on the number of edges of $G$.)

1.6.12 Let $G$ be a simple graph. Show that if $G$ is not connected then its complement $\bar{G}$ is connected.

1.6.13 A complete tripartite graph $G$ is a simple graph $G$ in which the vertex set $V$ is the union of three nonempty subsets $V_1$, $V_2$ and $V_3$ where $V_i \cap V_j = \emptyset$ for $i \neq j$ and an edge joins two vertices $u, v$ of $G$ if and only if $u$ and $v$ do not belong to the same $V_i$. If $V_1, V_2$ and $V_3$ have $r, s$ and $t$ elements respectively where $r \leq s \leq t$ we denote $G$ by $K_{r,s,t}$.

(a) Draw $K_{1,2,2}$, $K_{2,2,2}$, $K_{2,3,3}$.

(b) How many edges are there in $K_{r,s,t}$?

(c) Formulate a definition of a complete $n$-partite graph for any $n \geq 3$.

1.6.14 Which of the graphs in Figure 1.42 are bipartite? Justify your answer using Theorem 1.4 and redraw those that are bipartite showing the bipartite property more clearly.

![Graphs](image)

Figure 1.42: Which of these graphs are bipartite?

1.6.15 Let $G$ be a graph each of whose nonempty connected components is a bipartite graph. Assuming that $G$ has at least one nonempty component, prove that $G$ is bipartite. (This is used in the proof of Theorem 1.4 — see there on how to get started.)

Section 1.7. The Matrix Representation of Graphs

There are essentially two different ways of representing a graph inside a computer, namely by using the adjacency matrix or the incidence matrix of a graph.

Let $G$ be a graph with $n$ vertices, listed as $v_1, \ldots, v_n$. The adjacency matrix of $G$, with respect to this particular listing of the $n$ vertices of $G$, is the $n \times n$ matrix $A(G) = (a_{ij})$ where the $(i,j)$th entry $a_{ij}$ is the number of edges joining the vertex $v_i$ to the vertex $v_j$.

Figure 1.43 shows a graph $G$ with vertices listed as $v_1, \ldots, v_4$ and its adjacency matrix $A(G)$ with respect to this listing.

![Graph and Adjacency Matrix](image)

Figure 1.43: A graph and its adjacency matrix.

Note that in $A(G)$ we have $a_{ij} = a_{ji}$ for each $i$ and $j$. A matrix with this property is called symmetric. Note also that if $G$ has no loops then all the entries of the main diagonal of $A(G)$ are 0, while if $G$ has no parallel edges then the entries of $A(G)$ are either 0 or 1.

Given an $n \times n$ symmetric matrix $A = (a_{ij})$ in which all the entries are non-negative integers, we can associate with it a graph $G$ whose adjacency matrix is $A$, simply by letting $G$ have $n$ vertices, labelled 1 to $n$, say, and joining vertex $i$ to vertex $j$ by $a_{ij}$ edges. Figure 1.44 shows such a symmetric matrix $A$ and a graph produced from it.

![Symmetric Matrix and Graph](image)

Figure 1.44: A symmetric matrix $A$ of non-negative integers and a graph $G$ with $A(G) = A$. 

![Matrix and Graph](image)