CE 108 – HOMEWORK 5

EXERCISE 1. (GRADS)
Consider a stationary, zero-mean process, and assume that \( E[x(n)^2]=4 \) and \( E[x(n-1)x(n)]=3 \). Suppose you want to quantize this signal with 6 bits.

1. What is the theoretical SNR using PCM and using DPCM?

2. Repeat the exercise assuming \( E[x(n)^2]=4 \) and \( E[x(n-1)x(n)]=-1 \)

3. Repeat the exercise in 1. assuming that a zero-mean, white noise \( v(n) \) with variance \( \sigma_v^2=1 \) uncorrelated to \( x(n) \) is added to \( x(n) \). [“White” means that \( E[v(n-1)v(n)]=0 \)]

EXERCISE 2.
Predictive coding works by making a prediction, \( \hat{x}(n) \), of the next sample \( x(n) \), based on \( m \) previous samples: \( \hat{x}(n) = \sum_{k=1}^{m} \alpha_k x(n-k) \). When designing a predictor (i.e., when determining the coefficients \( \{\alpha_k\} \)), one should try to minimize the variance of the residual, \( e(n) = x(n) - \hat{x}(n) \), so that the residual can be encoded with few quantization levels.

1. For each one of the following coefficient sets, find and plot a signal \( x(n) \) such that the residual is always 0 (i.e., the signal is perfectly predicted from its past samples). Plot the signal only for \( n \geq 0 \), and assume that for \( n<0 \), \( x(n)=0 \). [Please avoid the trivial case \( x(n)=0 \)!]
   a. \( m=1, \alpha_1=1 \) (DPCM)
   b. \( m=2, \alpha_1=1, \alpha_2=1 \)
   c. \( m=2, \alpha_1=1, \alpha_2=-1 \)
   d. \( m=3, \alpha_1=1, \alpha_2=0, \alpha_3=-0.5 \)

2. A mismatch between the predictive model and the signal can create large prediction errors. What happens if a constant signal \( x(n)=1 \) is encoded using the predictive model d. in the previous question?

EXERCISE 3.
Suppose you are given a signal \( x(n) \) to encode with a predictive code. By looking at the signal, you infer that it slowly changes its statistical parameters over time. Hence you decide to divide the signal into segments \( S_1, S_2, \ldots, S_M \) of size \( N \) samples each, and for each one of these segments, you compute the optimal coefficients.
1. Prove that the sample variance of the residual of an optimal predictor over the union of two segments cannot be smaller than the average of the sample variances of the residuals of the optimal predictors for the two segments.

2. How can we determine the optimal size $N$ of a segment?

**EXERCISE 4. (GRADS)**
Consider the Wiener-Hopf (or Yule-Walker) linear system for the determination of the optimal coefficients of the predictor.
1. Prove that if the signal has mean equal to 0, then the residual $e(n)$ has mean equal to 0.

2. Prove that the residual $e(n)$ is uncorrelated to $x(n)$: $E[e(n)x(n)] = 0$

**EXERCISE 5.**
Plot the residual $e(n) = x(n) - \hat{x}(n)$ using Delta modulation with $\Delta=0.5$ and $\Delta=1$ for the following signals (assume that you start from $\hat{x}(n) = 0$):
1. $x(n) = n \mod 4$

2. $x(n) = 0.1$

**EXERCISE 6.**
Consider a transform coding system with two channels.

a. Intuitively, we can obtain a low value of the average distortion, $D_y$, if we can find a transformation $T$ that makes the variances in the two channels very different from each other. Can you give a formal proof of this intuition? (Hint: the sum of the variances in the two channels does not change as long as $T$ is orthonormal).

b. Give a proof of the fact that, under ideal optimal bit allocation, the variance of the error in the two channels is identical.

c. Suppose that the bit budget of the system is $B$, the input sampling rate is $R$, and the number of channels is $M$. What is the output bit rate?

d. Suppose you want to compare this system with a PCM system that simply quantizes each input sample with $B_{PCM}$ bits. What is the relationship between $B$ and $B_{PCM}$ if the output bit rate is the same in the transform coding and in the PCM system?
e. [GRADS] Let $\sigma^2_e$ be the variance of the quantization error in the PCM system and $D_x$ be the average distortion of the transform coding system. Prove that, under optimal bit allocation, and assuming that the transformation matrix $T$ is the Karhunen-Loeve transform, 

$$\frac{\sigma^2_e}{D_x} = \frac{\sigma^2_x}{(\det R_x)^{1/M}},$$

where $\sigma^2_x$ is the variance of the input signal and $R_x$ is its correlation matrix.

Suppose that the correlation of the transformed signal $y$ in the two channels, using a particular transformation $T$, is:

$$R_y = \begin{bmatrix} 32 & -3 \\ -3 & 2 \end{bmatrix}.$$ 

a. Can this transformation be the Karhunen-Loeve transformation?

Suppose the bit budget for this system is of 10 bits.

b. How would you allocate the bits in the two channels?

c. What is the resulting bit rate, assuming the input signal has sampling rate of 100 KHz?

d. Compute the resulting average distortion (noise) power and compare this with the average distortion power using PCM (under the same bit rate).