CE 108 – HOMEWORK 1

EXERCISE 1. (ALL)
Consider the sequence x(n) of values in
http://www.soe.ucsc.edu/classes/cmpe108/Fall07/ce108HW1.txt
http://www.soe.ucsc.edu/classes/cmpe108/Fall07/ce108HW1.xls.

1. If possible, plot the sequence (using any available software).
2. Determine the alphabet $S_x$ of the source and compute the probability of the symbols (based on their relative frequency in the sequence).
3. Determine the 1st order entropy of the source.
4. Transform the sequence $x(n)$ into the sequence $d(n)=x(n+1)-x(n)$. Determine the alphabet $S_d$, compute the symbol probabilities and the 1st order entropy for $d(n)$. Compare the latter with the 1st order entropy for $x(n)$.
5. Compute the 2nd order entropy of $d(n)$. To do so, determine the Cartesian product $S_d^2 = S_d \times S_d$, compute the probabilities of the symbols in $S_d^2$, and compute the corresponding entropy. Compare your result with the 1st order entropy of $d(n)$ and with the 2nd order entropy of $x(n)$. [Remember that the 2nd order entropy of a process is $H_2 = -1/2 \sum_x \sum_y P(x,y) \log_2 P(x,y)$, where $x$ and $y$ are two consecutive random variable in the process.]

EXERCISE 2. (GRADS - review)
Remember Jensen’s inequality: If $f(x)$ is convex, then $E[f(x)] \geq f(E[x])$. Use this to prove the following:
- $0 \leq H \leq \log_2 N$, where $H$ is the entropy of $x$ and $N$ is the size of the alphabet of $x$.
- $D_{KL}(P||Q) \geq 0$, where $D_{KL}(P||Q)=\sum_x P(x) \log(P(x)/Q(x))$ is the relative entropy (of Kullblak-Leibler divergence) of $P(x)$ and $Q(x)$ and $P(x)$, $Q(x)$ are mass distributions.

EXERCISE 3. (GRADS - review)
Use the results of Exercise 2. to prove that the 2nd order entropy of a process, $H_2$, is always $\leq$ the 1st order entropy $H$.

EXERCISE 4. (GRADS)
The “mutual information” of two variables is defined as $MI(x,y) = H(x)-H(x|y)$. Prove that $MI(x,y)$ satisfies the following properties:
- $MI(x,y) = MI(y,x)$
- $MI(x,y) \geq 0$

EXERCISE 5. (GRADS)
Consider the discrete random variables $x$ and $y=f(x)$, where $f(x)$ is a function.
- Prove that, if the $f(x)$ is bijective (invertible), then the entropy of $x$ is equal to the entropy of $y$.
- Prove that if $f(x)$ is not invertible, then the entropy of $x$ is larger than or equal to the entropy of $y$. 
Use the first result to prove that the entropy rate of a process does not change under invertible transformation of the process.

**EXERCISE 6. (UGRADS)**
Consider a variable $x$ with alphabet $A=\{x_1,x_2,x_3,x_4\}$ and probability mass distribution $\{P_1,P_2,P_3,P_4\}$, and a variable $y$ with alphabet $A=\{x_1,x_2,x_3\}$ and probability mass distribution $\{P_1,P_2,P_3+P_4\}$. Which one of the two has higher entropy?

**EXERCISE 7. (ALL)**
Consider a source that produces symbols from the alphabet $\{a,b,c,d\}$ at a rate of 5 symbols per second, with a probability distribution that changes cyclically in time. More precisely, the source alternates 1 second with probability distribution $\{0.1,0.15,0.3,0.45\}$ with 2 seconds with probability distribution $\{0.1,0.1,0.1,0.7\}$. Suppose we are using a code with lengths $\{3,3,2,1\}$. What is the average bit rate at the output of the coder?