April 14th - 1st midterm
May 10th - 2nd midterm

\[ A_1 \cup A_2 \cup \cdots \cup A_n = \Omega \]
\[ (\text{collectively exhaustive \& mutually exclusive}) \]
\[ P(\text{circle}) = \sum_{i=1}^{n} P(\text{piece of } A_i) \]
\[ \text{Law of total probability} \]

\[ A_1, \ldots, A_n \quad A_i \cap A_j = \emptyset \text{ if } i \neq j \text{ (mutually exclusive)} \]
\[ P(\text{A}_i \neq 0 \quad i=1, \ldots, n \quad (\text{not empty sets}) \]
\[ A_1 \cup A_2 \cup \cdots \cup A_n = \Omega \text{ (collectively exhaustive)} \]

\[ B_i = A_i \cap A_i \quad i=1, \ldots, n \]
\[ B_i \cap B_j = \emptyset \text{ if } i \neq j \]
\[ A = B_1 \cup B_2 \cup \cdots \cup B_n \]
\[ \therefore P(\text{A}) = \sum_{i=1}^{n} P(B_i) = \sum_{i=1}^{n} P(\text{A}_i \neq 0) P(\text{A} | A_i) \]

**Example:**

Airline has 5 distribution channels for sales:

<table>
<thead>
<tr>
<th>Channel</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20%</td>
</tr>
<tr>
<td>2</td>
<td>30%</td>
</tr>
<tr>
<td>3</td>
<td>10%</td>
</tr>
<tr>
<td>4</td>
<td>15%</td>
</tr>
<tr>
<td>5</td>
<td>25%</td>
</tr>
</tbody>
</table>

- 40% of tickets are profitable.

**What is probability that ticket is profitable?**

**Event A:**

\[ A = \{ \text{sale is profitable} \} \]
\[ A_i = \{ \text{sale of any type of } \} \text{ in channel } i \}

\[ P(\text{A}) = P(\text{A}_1 \neq 0) P(\text{A} | A_1) + \cdots + P(\text{A}_5 \neq 0) P(\text{A} | A_5) \]

\[ = 0.2 \times 0.4 + 0.3 \times 0.6 + 0.1 \times 0.2 + 0.15 \times 0.5 + 0.25 \times 0.9 \]
A, B in same sample space
Events A and B are said to be independent, if \( P(A \cap B) = P(A)P(B) \).

\[ P(A \cap B) = P(A)P(B) \]

\( P(A)P(B) \) means A does not matter if A, B are independent.

A, B independent but does not imply A, C independent.

A, B independent \( \Rightarrow \overline{A}, \overline{B} \) and \( A, \overline{B} \) and \( \overline{A}, B \) are independent.

\[ (A \cap B) \cup (\overline{A} \cap B) = B \]

\[ P(\overline{B}) = P(A \cap B) + P(\overline{A} \cap B) \]

\[ = P(B)P(A) + P(\overline{A})P(B) \]

\[ P(\overline{A} \cap B) = P(\overline{B})(1 - P(A)) \]

\[ = P(B)P(A) \]

A\(_1\), ..., A\(_n\) are mutually independent.

\[ P(A_i \cap A_j) = P(A_i)P(A_j) \quad i \neq j \]

\[ P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k) \quad i \neq j \neq k \]

\[ P(\bigcap_{i=1}^{n} \overline{A_i}) = \prod_{i=1}^{n} P(A_i) \]

Most of the time, we assume independence.

\( S_6 = \{(i, j) \mid i, j \in [1, 6]\} \) Each sample point has probability \( \frac{1}{36} \).

Event A = die 1 results in 1, 2, 3
Event B = die 1 results in 3, 4, 5
Event C = sum of the two faces = 9

continued on next page
\[
A \cap B = \{(3,1), (3,2), \ldots, (3,6)\}
\]
\[
A \cap C = \{(3,6)\}
\]
\[
B \cap C = \{(3,6), (4,5), (5,4)\}
\]
\[
A \cap B \cap C = \{(3,6)\} = \frac{1}{36}
\]
\[
P(E\cap A_3) = \frac{1}{2}, \ P(E\cap B_3) = \frac{1}{2}, \ P(E\cap C_3) = \frac{1}{9}.
\]
\[
P(A_3|E\cap B_3 \cap C_3) = \frac{1}{36}
\]
\[
P(A_3|E\cap C_3) = \frac{1}{18} \neq \frac{1}{36}
\]
80% of programs are written in C.

Only 20% of C programs compile on 1st attempt.

E program compiles ok.

Program compiled on 1st attempt. What is the likelihood that the program is written in C? \( P(C|E) \)

\[
P(C) = 0.8 \quad P(E|C)_3 = 0.2
\]
\[
P(J) = 0.2 \quad P(E|J)_3 = 0.6
\]
\[
P(E \cap C \cap J) = P(E \cap C \cap E) = \frac{P(E \cap C_3 \cap E_3)}{P(E_3)} = \frac{P(E \cap C_3 \cap E_3)}{P(E_3) \cdot P(E \cap C_3 \cap E_3) + P(J \cap E \cap I \cap J_3)}
\]

\[\text{Bayes' Theorem}\]

A probability of cause by symptom

A_1, \ldots, A_n \text{ mutually exclusive & collectively exhaustive}

\[
P(E \cap A_3) = P(E \cap A_3) \cdot P(E \cap A_3)
\]

\[\text{Bayes' Theorem}\]

A priori events

Series & parallel systems

Series

Parallel

Series = \( P(E \cap A_1 \cap \ldots \cap A_n) \) all components must be ok

Reliability \( R \)

Assumption: Failure of components are mutually exclusive events.

A_i component i is OK

By definition, reliability of component i = \( \frac{R_i}{\prod_{i=1}^{n} R_i} \)
1 component \( R_1 = 0.98 \)  
5 \( R_5 = 0.904 \)  
10 \( R_{10} = 0.817 \)

Reliability of parallel system of \( n \) components

\[ R_p = \prod_{i=1}^{n} \bar{A}_i \]

\[ \bar{A}_p = \text{system failed} \]

\[ \bar{A}_p = \text{failed components} \]

\[ P\{\bar{A}_p\} = P\{\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n\} \]

\[ = P\{\bar{A}_1 \} \cdot P\{\bar{A}_2 \} \cdot \ldots \cdot P\{\bar{A}_n \} \]

Reliability of system

\[ = 1 - \prod_{i=1}^{n} P\{\bar{A}_i \} \]

\[ = 1 - \left(1 - R_i\right)^n \]

Reliability of system

Series-Parallel System: \( n \) serial stages \( i \) replicated \( n_i \) times

\[ R_{sp} = \prod_{i=1}^{n} \left[1 - (1 - R_i)^{n_i}\right] \]