Bayes Theorem

Assume Partition of sample space into mutually exclusive and collectively exhaustive events $A_1, ..., A_n$.

Let $A$ be an event.

Then, $\frac{\prod_{i=1}^{n} P(A_i \cap A) P(A_i | A)}{\sum_{j=1}^{n} P(A_j \cap A) P(A_j | A)} = P(A_i | A)$

$P(A_i)$ a priori - Assume you know these.

$P(A_i | A)$ a posteriori - "Knowing what has happened"

Event $D = \text{member of class having disease}$

Event $S = \text{test result indicates the disease}$

Sensitivity of test $= \gamma = P(S \mid D)$ probability of identifying a defect given a defect

Specificity of test $= \theta = P(S \mid \overline{D})$

$\cup \overline{D} = D \cup \overline{D}$ defect union no defect

$P(D \mid S) = \text{predictive value of positive test}$

$P(D)$ is small $\Rightarrow$ you get a large # of false positive
False positive = \( P(\bar{D}|\bar{E}) = 1 - P(E|\bar{D}) \bar{E} = 1 - \theta \)

**Example** \( \pi = P(E|D) \) there is a defect.

\[ \pi = 0.0001 \text{ infection ratio} \]

Assume \( \gamma = 0.977 \)
\( \theta = 0.926 \)

Create 100,000 samples/tests

100,000 \( \times \) 0.0001 \( \Rightarrow \) 10 actually infected

9.77 correctly diagnosed

99,990 don't have disease

1-\( \theta \) = 0.075

99,990 \( \times \) (1-\( \theta \)) = rate of false positives

\( \Rightarrow \) 7,399.20 incorrectly told that they have disease

**Example**

binary channel

0,1

Noise in channel 0 \( \Rightarrow 0.94 \) no error in channel

1 \( \Rightarrow 1.091 \) no error from sending 1

Prob of transmitting "0" = 0.45

Signal sent,

Determine prob that \( 0 \) is received \( R_0 \)

\( \hat{0} \) is received \( R_0 \)

\( \hat{1} \) was sent given that \( \hat{1} \) was received \( T_1|R_1 \)
4. $O$ was sent given $D$ was received $T_o \mid R_o$

5. Probability of error $(T_1 \wedge R_0) \cup (T_0 \wedge R_1)$

$T_0 = 0$ was sent (transmitted)

$R_0 = 0$ was received

$T_1 = 1$ was sent $\overline{T_1} = \overline{T_0}$

$R_1 = 1$ was received $\overline{R_1} = \overline{R_0}$

$0.45 = P_{T_0}^2$

$0.94 = P_{R_0 \mid T_0}^3$

$0.91 = P_{R_1 \mid T_1}^3$

$P_{ER_0}^3 = P_{ER_0 \wedge T_1}^3 + P_{ER_0 \wedge T_0}^3$ can add because they are mutually exclusive

$P_{ET_1}^3 = P_{ER_0 \wedge T_1}^3 P_{ET_1}^3 + P_{ET_0}^3 P_{ER_0 \wedge T_0}^3$

$1 - P_{ET_0}^3$

$= 0.55$

Note: $P_{ER_0 \wedge T_1}^3 = 1 - P_{ER_1 \wedge T_1}^3 = 0.09$

$\therefore P_{ER_1}^3 = 1 - P_{SR_0}^3$

$P_{ET_1 \wedge R_1}^3 = \frac{P_{ER_1 \wedge T_1}^3 P_{ET_1}^3}{P_{ER_1}^3}$

$P_{ET_0 \wedge R_0}^3 = \frac{P_{ET_0}^3 P_{ER_0 \wedge T_0}^3}{P_{ER_0}^3}$
$$P \{ \text{error} \} = P \{ T_i \cap R_o \} \cup (T_o \cap R_i \}$$

These two events are mutually exclusive.

$$= P \{ T_i \cap R_o \} + P \{ T_o \cap R_i \}$$

$$= P \{ R_o \} P \{ T_i | R_o \} + P \{ R_i \} P \{ T_o | R_i \}$$

$$= \prod P \{ R_o \} P \{ T_i | R_o \} \prod P \{ T_i \} + P \{ T_o \} P \{ R_i | T_o \}$$

Reliability of Systems. Series parallel systems

\[ A \rightarrow R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow B \]

Series = system fails if any fails.

Parallel arrangement: if system fails, all must fail.

Consider series \( n \) components

Assume \( A_i \): component \( i \) is OK

Reliability of component \( i \) is \( R_i = P \{ A_i \} \).

Failure of components are mutually independent events.

\[ P \{ A_1 \cap A_2 \cap \ldots \cap A_n \} = \text{reliability of system} \]

\[ \prod P \{ A_i \} = \prod R_i \]

Product law of reliability.

0.98 = reliable of each component

\[ 5 \rightarrow 0.904 \]

10 \rightarrow 0.817

20% failure.

To achieve higher reliability, people introduced redundancy.
redundancy

- parallel system of n components
- A_p functions OK

\[ P \bar{\Sigma} \bar{A}_p \bar{\Sigma} = \text{all (ten) fail} = P \bar{\Sigma} \bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n \bar{\Sigma} \]

\[ P \bar{\Sigma} A_p \bar{\Sigma} \text{ hard to compute but fails are indep. but nonfailures are indep.} \]

\[ = P \bar{\Sigma} \bar{A}_1 \bar{\Sigma} P \bar{\Sigma} \bar{A}_2 \bar{\Sigma} \ldots P \bar{\Sigma} \bar{A}_n \bar{\Sigma} \text{ each fail} \]

\[ F_i = 1 - R_i \quad \text{unreliability} \quad F = \prod_{i=1}^{n} F_i \]

\text{unreliability of whole system}

Either each has to be reliable or you need a high degree of redundancy

Most systems are a mix of series and parallel.

Have a series system but want reliability:

- \begin{array}{c}
\text{duplicated} \\
\text{replicated}
\end{array}

"series-parallel system"

Stage i A_i replicated \[ R_{sp} = \prod_{i=1}^{n} \left[ 1 - (1 - R_i)^{n_i} \right] \]

Try to derive this \( \rightarrow \)
Example:

\[
\begin{align*}
A_i &= \text{component } i \text{ is OK} \\
R_i &= \Pr[A_i] \quad i = 1, \ldots, 5 \\
A &= \text{system is OK} \\
R &= \Pr[A] \\
A &= (A_1 \land A_4) \lor (A_2 \land A_4) \lor (A_2 \land A_5) \lor (A_3 \land A_5) \\
&\text{Note events are not mutually exclusive!} \\
&\text{Look @ structure of system & apply law of total probability} \\
\Pr[A] &= \Pr[A_2] \Pr[A \mid A_2] + \Pr[A_2] \Pr[A \mid A_2] \\
&\text{if component 2 fails:} \\
A &= \begin{array}{c}
\rightarrow \ C_2
\end{array} \rightarrow \begin{array}{c}
\rightarrow \ C_4
\end{array} \\
&\Pr[A \mid A_2] = \Pr[A_4 \lor A_5] \\
&= 1 - (1 - R_4) \times (1 - R_5) \\
&\text{mutually independent failures}
\end{align*}
\]
\[ P\{A | \bar{A}_2 \} = 1 - (1-R_1 R_4)^* (1-R_3 R_5) \]

reliability of \( C_1 - C_4 = R_1 R_4 \)

\( C_3 - C_5 = R_3 R_5 \)

Next is Random variables and how you characterize them.