Midterm a week from Thursday.

Homework Set 2: Due next Tuesday

44  #1,3
45  #18
52  #16
82  #2
87  #1,4,7
97  #8,11
106 #3,7,9
119 #8

Definition: Conditional prob of event $A$ given event $B$

$$\Pr \{ A \mid B \} = \frac{\Pr \{ A \cap B \}}{\Pr \{ B \}}$$

Joint probability of events $A$ and $B$

Nontrivial

$\Pr \{ B \mid B \} = 1$

Now everything is conditioned on event $B$

$$\therefore \Pr \{ A \cap B \} = \Pr \{ A \} \Pr \{ B \mid A \}$$

Multiplication rule

$$\Pr \{ A \cap B \} = \Pr \{ B \} \Pr \{ A \mid B \}$$
General multiplication rule:

\[ P \{ A_1 \cap A_2 \cap \ldots \cap A_n \} = \]

\[ = P \{ A_1 \} \cdot P \{ A_2 \mid A_1 \} \cdot P \{ A_3 \mid A_1, A_2 \} \ldots P \{ A_n \mid A_1, A_2, \ldots, A_{n-1} \} \]

Sufficient but not necessary that

\[ P \{ A_1 \cap \ldots \cap A_{n-1} \} > 0 \]

Example: 100 accounts (installations)

\[ \frac{75}{100} \] installations have at least one computer of that brand.

Your assignment is to visit 3 installations today?

What's probability that the computer brand is yours?

\[ A_1 = \text{event that 1st install is customer} \]

\[ A_2 = \text{2nd} \]

\[ A_3 = \text{3rd} \]

\[ P \{ A_1 \cap A_2 \cap A_3 \} = \]

\[ = P \{ A_1 \} \cdot P \{ A_2 \mid A_1 \} \cdot P \{ A_3 \mid A_1, A_2 \} \]

\[ = \frac{75}{100} \cdot \frac{74}{99} \cdot \frac{73}{98} \]

\[ \approx 0.42 \]
Example.

5000 memory chips \(\leftarrow 4000\) brand \(y\) (5% defects)
\(\leftarrow 1000\) brand \(x\) \(\{10\%\) defects\}

\[\begin{align*}
A &= \{\text{chip is of brand } x\} \\
B &= \{\text{chip is defective}\}
\end{align*}\]

\[\begin{align*}
P\{A\} &= \frac{1000}{5000} = 0.2 \quad & P\{B\} &= \frac{300}{5000} = 0.06 \\
&= 0.32
\end{align*}\]

Ask: \(P(A \cap B) = ?\)

\[\begin{align*}
P\{A \cap B\} &= \frac{100}{5000} = 0.02
\end{align*}\]

Select at random & it is defective,
P(chip is manufactured by \(x\)) \(\equiv P\{A|B\}\)

\[\begin{align*}
P\{A|B\} &= \frac{P\{A \cap B\}}{P\{B\}} \\
&= \frac{0.02}{0.06} \\
&= 0.3333
\end{align*}\]

Two events \(A, B\) assuming
\(P\{A|B\} \neq 0\)
\(P\{B|A\} \neq 0\).

\[\begin{align*}
P\{A|B\} &= \frac{P\{A \cap B\}}{P\{B\}} \\
&= \frac{P\{A\cap B\}}{P\{A\}} \cdot \frac{P\{A\}}{P\{B\}} \\
&= P\{A\} \quad \text{when} \quad P\{B\} \neq 0
\end{align*}\]
circle is event \( A \)

4 mutually exclusive pieces totally partition sample space \( \Omega \).

Law of total probability:

\[
\bigwedge
\text{set of events } A_0 \ldots A_n
\]

(a) such that \( A_i \cap A_j = \emptyset \) if \( i \neq j \) (mutually exclusive)

(b) \( P(A_i) > 0 \quad \forall \ i = 1, \ldots, n \)

(c) \( A_1 \cup \ldots \cup A_n = \Omega \) those pieces are a family of events satisfying (a)–(c) collectively exhaustive that is a partition of \( \Omega \).

Theorem:

For any event \( A \),

\[
P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + \ldots + P(A_n)P(A|A_n)
\]

Let \( B_i = A \cap A_i, \ i = 1, \ldots, n \).

\( B_i \cap B_j = \emptyset \ \forall \ i \neq j \)

\( A = B_1 \cup B_2 \cup \ldots \cup B_n \quad \Rightarrow \quad P(A) = P(B_1) + \ldots + P(B_n) \)

\( \text{q.e.d.} \)
Example.

An airline sells tickets through 5 channels.

Channel 1 2 3 4 5

20% 80% 10% 15% 25%

0.4 0.6 0.2 0.8 0.9

What is probability that airline has profitable sale?

Define A, profitable sale

A_i, sale happens on channel i

\[ P(A_3) = P(A_3 | A_1) P(A_1) + P(A_2) P(A_1 | A_2) + P(A_5) P(A_1 | A_5) \]

\[ P(A) = (0.2)(0.4) + (0.3)(0.6) + 0.15 \times 0.2 \]
\[ + 0.15 \times 0.8 + 0.2 \times 0.9 \]

Two events A and B are independent if \( P(A \cap B) = P(A) P(B) \), it means:

\[ P(A \cap B) = P(A) P(B | A) = P(B) P(A | B) \]

assuming neither event is impossible.

\[ P(A_3) \]
nontransitive:  \( A, B, C \)
\[ A, B \text{ independent} \]
\[ B, C \text{ independent} \not\Rightarrow A, C \text{ independent} \]

however

if \( A \) and \( B \) are independent
\[ \overline{A} \text{ and } \overline{B} \]
\[ \overline{A} \text{ and } B \]
\[ \overline{A} \text{ and } \overline{B} \]

\[ \text{Proof: } A \cap B, \overline{A} \cap B \text{ mutually exclusive} \]
\[ \bar{\text{U}} = B \]

\[ P\{B\} = P\{A \cap B\} + P\{\overline{A} \cap B\} = P\{A\} P\{B\} + P\{\overline{A}\} P\{B\} \]

\[ P\{\overline{A} \cap B\} = P\{B\}[1 - P\{A\}] = P\{B\} P\{\overline{A}\} \]
\[ \text{if } A, B \text{ independent} \]

Set of \( n \) events mutually independent,
\[ P(\text{set}) = \text{product of } n \text{ events. in any combination} \]

\[ A_1, \ldots, A_n \text{ are mutually independent, \text{ \text{distinct}} if and only if for any set of } k \text{ indices } \]
\[ (k \in \{2, n\}) \]
\[ i_1, \ldots, i_k \]

\[ P\{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}\} = P\{A_{i_1}\} P\{A_{i_2}\} \ldots P\{A_{i_k}\} \]
\[ \text{they are mutually independent} \]
tossing 2 dice \( \Omega = \{ (i, j) : i, j \in \{1, 2, 3, 4, 5, 6\} \} \)

Each sample point has prob \( \frac{1}{36} \).
die 1 results in 1, 2, 3 = event A \( P(A) = \frac{1}{3} \)
die 1 results in 3, 4, 5 = event B \( P(B) = \frac{1}{2} \)
sum of 2 faces = 9 = event \( C = \{(3,6), (4,5) \}
\( P(C) = \frac{1}{9} \).

Observe A, B, C are mutually independent.\)

\[ A \cap B = \{(3,1), (3,2), \ldots, (3,6) \} \]
\[ A \cap C = \{(3,6) \} \]
\[ B \cap C = \{(3,6), (4,5), (5,4), (6,3) \} \]
\[ A \cap B \cap C = \{(3,6) \} \]
\[ P(A \cap B \cap C) = \frac{1}{36} \]

\[ P(A) \cdot P(B) \cdot P(C) = \frac{1}{36} \]

So they might be independent.\)

\[ P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{1}{9} \]

\[ P(A \cap B) = \frac{1}{6}, \quad P(A \cap B \cap C) = \frac{1}{4} \]

not independent.\)

Note:\)
For writing a program,
2 languages are allowed:
80% in "C"
20% in "J"
but only 20% of routines are correct on 1st attempt.
A randomly selected routine was correct on 1st attempt. How likely written in C language?

E program O.K.
C written in "C"
J written in "J"

\[ P(C|E) = ? \]
\[ = \frac{P(C \cap E)}{P(E)} = \frac{P(C|E)cE} {P(E)} = \frac{P(C|E)cE}{P(E)} + P(J|E)cE \text{ since } J \& C \text{ partition } \Sigma \]
\[ = \frac{P(cE|C)cE}{P(E)} \]
\[ = \frac{0.8 \times 0.2}{0.20} \]
\[ = 0.80 \]