\( \Omega \), sample space

\[ X = \text{some function of each sample point in } \Omega \]

\[ X(\omega) = \text{value it takes on} \]

" \( X = x \) " is \( \{ \omega : \omega \in \Omega \text{ and } X(\omega) = x \} \)

" \( X \leq x \) " is \( \{ \omega : \omega \in \Omega \text{ and } X(\omega) \leq x \} \)

" \( y < X \leq x \) " is \( \{ \omega : \omega \in \Omega \text{ and } y < X(\omega) \leq x \} \)

\( X \leq x \quad \forall x \) must make sense, has to be a measurable function

Cumulative distribution function = CDF

\[ F(x) = P \{ X \leq x \} \text{ defined for all real } x \]

Properties of \( F(\cdot) \):

1. Non-decreasing function of \( x \)

\[ x < y \Rightarrow F(x) \leq F(y) \]

2. \( \lim_{x \to \infty} F(x) = 1 \)

3. \( \lim_{x \to -\infty} F(x) = 0 \)

"Complete description of a R.V."
\( X \), Random Variable

Claim:
\[ P\{x < X \leq y\} = F(y) - F(x) , \]

\( E_1 = \{x < X \leq y\} \), \( E_2 = \{ X \leq x\} \) are disjoint.

union of \( E_1 \) and \( E_2 = \{ X \leq y\} \)

\[ \therefore P\{X \leq y\} = P\{x < X \leq y\} + P\{X \leq x\} \text{ because they are disjoint}. \]

\[ F(x) \]

\[ F(y) \]

\[ \text{q.e.d.} \]

C.D.F. cumulates everything - do not exceed that point.

Given \( X \), we can define pmf "probability mass function"
\[ p(x) = P\{X = x\} \hspace{1cm} \text{For continuous R.V. it identically } 0 \]
\[ \text{at any point only} \]
\[ \text{i.e. pmf at sum of 2 faces is } 9 \]
\[ \text{For any discrete R.V.} \]

Properties
1. \( p(x) > 0 \ \forall x \)

2. \( T = \{ x : p(x) > 0 \} \) is finite or countably infinite

3. \[ \sum_{x \in T} p(x) = 1 . \]

\[ p(x) > 0 \iff \text{"mass points"} \]
Recall this example:

R.V. \( X \)
- 5 polls: \( 0.0 \) are ready
- 1 \( p(1) = 0.6 \)
- 2 \( p(2) = 0.3 \) ~ computational pmf
- 3 \( p(1) = 0.1 \)

For continuous R.V.:

1. \( f(x) \geq 0 \) \( \forall x \) ~ "not a probability"
2. \( f(\cdot) \) integrable
   \[ P \{a \leq X \leq b\} = \int_a^b f(x) \, dx \]
3. \( F(x) = \int_{-\infty}^x f(t) \, dt \) \( \forall x \in \mathbb{R} \)

4. At each point that density \( f(x) \) is continuous,
   you have that \( f(x) = \frac{d}{dx} F(x) \)
\[ f(x) \, dx \text{ interpret as } P\{x \leq X < x + dx\} \]

\( X, R.V. \text{ continuous} \)
\[ P\{a \leq X < b\} = P\{a < X \leq b\} = P\{a \leq X < b\} \]
\[ = P\{a < X < b\} \]

\( \text{If I have discrete RV \& I have pmf} \)
\( \text{Then I can compute everything.} \)

PMF \quad PDF \quad CDF

A "complete characterization" = "how likely" is computable
also can measure mean \& expected response time

R.V. \( X \) \quad \( h(X) = 2X \)
\quad or \( e^X \)
\quad or...

\( h(X) \) is a new R.V.,

Expected value of \( h(X) \equiv E[\ h(X)\ ] \)

If \( X \) is discrete R.V., then \( E = \sum_{i} h(x_i) p(x_i) \) \( \text{sum mass point} \)

\begin{array}{c|c|c}
 x_i & p(x_i) & h(x_i) \\
 \hline
 1 & 0.6 & \text{response time} \\
 2 & 0.3 & 2 \text{ ms} \\
 3 & 0.1 & 10 \text{ ms} \ \\
 & & 100 \text{ ms} \\
\end{array}
Compute expected C.P.U. time = $2 \times 0.6 + 10 \times 0.3 + 100 \times 0.1$

If $X$, R.V. is continuous, then $E = \int_{-\infty}^{+\infty} h(x)f(x)\,dx$.

"Expected value of function of R.V. = weighted average or "mean"

\[ \mu = E[X] \]

mean

if $X$ R.V. continuous, $\mu = \int_{-\infty}^{+\infty} x\cdot f(x)\,dx$

\[ \mu = \sum_{i} x_i \cdot p(x_i) \] 

discrete

Throw 2 fair dice, what's expected outcome?

$\mu = \sum_{n=2}^{12} n \cdot p(n)$, if fair $p(n) = \frac{1}{2}$ \cdot \frac{n}{2}.

Example

Variance $\sigma^2 = Var[X]$

\[ Var[X] = E[(X - E[X])^2] \]

$\sqrt{Var[X]} = \text{standard deviation, } \sigma$

If R.V. is discrete $\sigma^2 = \sum_{X_i} (x_i - E[X])^2 p(x_i)$. 

sum over all mass points
If $X$ is continuous
\[
\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) \, dx
\]

Example
50% $\$2K  \quad \text{Clearly a pmf}
50% $\$0.02K  \quad \leftarrow 2 \text{ mass points}

Mean income = $\mu = 0.5 \times 2 + 0.5 \times (0.02) = 1.01$

\[
\sigma^2 = \sum_{x_i} (x_i - \mu)^2 \times p(x_i)
\]
\[
= (0.02 - 1.01)^2 \times 0.5 + (2 - 1.01)^2 \times 0.5
= 0.9801
\]

$\sigma = 0.99$

$\mu$ & $\sigma$ are similar; $\mu$ is not meaningful!