Thursday, Nov 4

\[ \Psi(e^\theta) = g(e^\theta) \]
\[ g(z) = \Psi(\ln z) \]

Let \( Y = aX + b \);

\[ \Psi_y(\theta) = E[e^{\gamma Y}] = E[e^{e^{\gamma X}e^{\gamma b}}] = e^{\gamma b} E[e^{\gamma X}] \]
\[ = \Psi_x(a\theta) \]

**Inequalités**

Assume \( E[X] < \infty \), can we say anything about \( P\{X > t\} \)?

Consider a discrete R.V.

\[ E[X] = \sum_{x_i} x_i p(x_i) = \sum_{x_i < t} x_i p(x_i) + \sum_{x_i \geq t} x_i p(x_i) \]
\[ \geq \sum_{x_i \geq t} x_i p(x_i) \overset{\text{smallest value it could be}}{=} \sum_{x_i \geq t} t p(x_i) \]
\[ = t \sum_{x_i \geq t} p(x_i) = t \cdot P\{X > t\} \]
\[ P\{X > t\} \leq \frac{E[X]}{t} \], Markov's inequality.

10s = to retrieve data from some archival media
\[ P\{\text{response time} \geq 20s\} \leq \frac{10}{20} = 0.5 \]

\[ P\{X > kE[X]\} \leq \frac{E[X]}{kE[X]} = \frac{1}{k} \]

X assume \( E[X] < \infty \)
\( \sigma^2 = \text{Var}[X] \) known \( \text{I know the 1st two moments} \)

\((X - E[X])^2\)
\( t^2 \) is target value

\[ \{P\{(X - E[X])^2 \geq t^2\} \leq \frac{E[(X - E[X])^2]}{t^2} = \frac{\sigma^2}{t^2} \]

\((X - E[X])^2 \geq t^2 \iff |X - E[X]| > t \)

\[
\therefore \quad P\{|X - E[X]| > t\} \leq \frac{\sigma^2}{t^2} \quad \text{Chebychev's inequality}
\]
optical retrieval \(40s\)

\[ \sigma = 2s \]

retrieval time should be \([4, 16s]\) within this

\[ P\{T \leq 4\} \cup P\{T \geq 16\} \leq \frac{4}{6^2} = \frac{1}{9} \]

\[ P\{4 < T < 16\} \geq \frac{8}{9} \]

\(X\) - exponential R.V. with mean \(E[X] = 2\).

\[ \lambda = \frac{1}{2} \]

\[ P\{|X - E[X]| > 4\} = P\{|X - 2| > 4\} \]

\[ = 1 - P\{X \leq 6\} \]

\[ = 1 - (1 - e^{-6/2}) = e^{-3} = 0.0498 \]

\[ P\{|X - E[X]| > 4\} \leq \frac{4}{16} = 0.25 \]

New example
\([-2, 0, 2\] \(\sim X\)

\[ E[X] = 0 \]

\[ \text{prob. } \frac{1}{8}, \frac{3}{4}, \frac{1}{8} \]

\[ E[X^2] = 1 \]

\[ \text{Var}[X] = 1 \]

\[ P\{|X - E[X]| > 2\} \leq \frac{4}{4} \]

\(X\) deviates from mean
Given that $X, \ E[X], \ \sigma^2 < \infty$,

- \( P\{X \leq t\} \leq \frac{\sigma^2}{\sigma^2 + (t - E[X])^2} \) if \( t < E[X] \),
- \( P\{X > t\} \leq \frac{\sigma^2}{\sigma^2 + (t - E[X])^2} \) if \( t > E[X] \).

Let $T = 400$ ms, $E[T] = 400$ ms, $\sigma = 116$ ms.

Is the device worthwhile building?

Chebyshev's inequality requires

\[ P\left\{ T \leq 750 \right\} \geq 0.9 \]

\[ 0.9 \] \text{ of retrievals must not exceed 750 ms.}

\[ P\left\{ T > 750 \right\} = P\left\{ T - 400 > 350 \right\} \]

\[ \leq P\left\{ \frac{|T - 400|}{350} > 1 \right\} \]

\[ \leq \left( \frac{116^2}{350^2} \right) = 0.109 \]

Need to have it \( \leq 1 \) to satisfy design.
So, we try the one-sided inequality:
\[ P \{ T > 750 \} \leq \frac{(116)^2}{(116)^2 + (750 - 400)^2} = 0.099 \]

\[ \Rightarrow P \{ T \leq 750 \} \geq 0.901 . \]

- one-sided inequality is a sharper bound.

New
Event A
probability of an event
n trials
count # times, \( S_n \), that A occurs.
\[ \frac{S_n}{n} \sim P \{ A \} \]
Laws of large numbers.

A, event \( P \{ A \} \)
sequence of \( n \) Bernoulli trials
Success is: A occurs.

\( S_n = \# \) of successes in \( n \) trials
\( S_n \) has binomial distribution
\( p = P \{ A \} \)
\[ E[S_n] = np \]
\[ q = 1 - p \]
\[ = n \cdot P \{ A \} \]
\[ Var[S_n] = npq = nP \{ A \} (1 - P \{ A \}) . \]
Actually use
\[
\frac{S_n}{n}
\]
to estimate measurement

\[
E \left[ \frac{S_n}{n} \right] = \frac{1}{n} E[S_n]
\]

\[= \text{PEA}3\]

\[
\text{Var} \left[ \frac{S_n}{n} \right] = \frac{1}{n^2} \text{Var}[S_n] = \frac{\text{PEA}3(1-\text{PEA}3)}{n}
\]

By Chebyshev's inequality,

\[
Pr \left| \frac{S_n}{n} - \text{PEA}3 \right| \geq \varepsilon \leq \frac{\text{PEA}3(1-\text{PEA}3)}{n \varepsilon^2} \leq \frac{a_{25}}{n \varepsilon^2}
\]

"I want to be 95\% sure that I am within 0.5\% of the true value"

\[
\text{PEA}3(1-\text{PEA}3) \leq 0.25
\]

\[0.5 \quad p\]