Problem 5.1.2

(a) Because the probability that any random variable is less than $-\infty$ is zero, we have

$$F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0$$

(b) The probability that any random variable is less than infinity is always one.

$$F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x)$$

(c) Although $P[Y \leq \infty] = 1$, $P[X \leq -\infty] = 0$. Therefore the following is true.

$$F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0$$

(d) Part (d) follows the same logic as that of part (a).

$$F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0$$

(e) Analogous to Part (b), we find that

$$F_{X,Y}(y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y)$$

Problem 5.2.1

(a) The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

To find the constant $c$ we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} cy \, dy \, dx = cx - \frac{cx^2}{2} \bigg|_0^1 = \frac{c}{2} = 1$$

Therefore $c = 2$. 

Y + X = 1
(b) To find the \( P[X \leq Y] \) we look to integrate over the area indicated by the graph

\[
P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dydx
= \int_0^{1/2} (2-4x)dx
= 1/2
\]

(c) The probability \( P[X+Y \leq 1/2] \) can be seen in the figure at right. Here we can set up the following integrals

\[
P[X+Y \leq 1/2] = \int_0^{1/2} \int_0^{\sqrt{1/2-x}} dydx
= \int_0^{1/2} (1-2x)dx
= 1/2 - 1/4 = 1/4
\]

**Problem 5.3.1**

(a) The joint PDF (and the corresponding region of nonzero probability) are

\[
f_{X,Y}(x,y) = \begin{cases} 
  1/2 & -1 \leq x \leq y \leq 1 \\
  0 & \text{otherwise}
\end{cases}
\]

(b)

\[
P[X > 0] = \int_0^1 \int_x^{1/2} dydx = \int_0^1 \frac{1-x}{2} dx = 1/4
\]

This result can be deduced by geometry. The shaded triangle of the \( X,Y \) plane corresponding to the event \( X > 0 \) is 1/4 of the total shaded area.

(c) For \( x > 1 \) or \( x < -1 \), \( f_X(x) = 0 \). For \(-1 \leq x \leq 1\),

\[
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_x^{1} \frac{1}{2} dy = (1-x)/2
\]

The complete expression for the marginal PDF is

\[
f_X(x) = \begin{cases} 
  (1-x)/2 & -1 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}
\]
From the marginal PDF \( f_X(x) \), the expected value of \( X \) is

\[
E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \frac{1}{2} \int_{-1}^{1} x(1-x) \, dx = \frac{x^2}{4} - \frac{x^3}{6} \bigg|_{-1}^{1} = -\frac{1}{3}
\]

**Problem 5.3.2**

\[
f_{X,Y}(x,y) = \begin{cases} 
2 & x+y \leq 1, x, y \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions.

\[
f_X(x) = \int_{0}^{1-x} 2 \, dy = \begin{cases} 
2(1-x) & 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Likewise for \( f_Y(y) \):

\[
f_Y(y) = \int_{0}^{1-y} 2 \, dx = \begin{cases} 
2(1-y) & 0 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Problem 5.5.1**

\[
f_{X,Y}(x,y) = \begin{cases} 
(x+y)/3 & 0 \leq x \leq 1, 0 \leq y \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

Before calculating moments, we first find the marginal PDFs of \( X \) and \( Y \). For \( 0 \leq x \leq 1 \),

\[
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_{0}^{2} \frac{x+y}{3} \, dy = \frac{xy}{3} + \frac{y^2}{6} \bigg|_{y=0}^{y=1} = \frac{2x+2}{3}
\]

For \( 0 \leq y \leq 2 \),

\[
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx = \int_{0}^{1} \frac{x+y}{3} \, dx = \frac{x^2}{6} + \frac{xy}{3} \bigg|_{x=0}^{x=1} = \frac{2y+1}{6}
\]

Complete expressions for the marginal PDFs are

\[
f_X(x) = \begin{cases} 
\frac{2x+2}{3} & 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases} \quad f_Y(y) = \begin{cases} 
\frac{2y+1}{6} & 0 \leq y \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

(a) The expected value of \( X \) is

\[
E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{0}^{1} \frac{2x+2}{3} \, dx = \frac{2x^2}{9} + \frac{x^3}{3} \bigg|_{0}^{1} = \frac{5}{9}
\]

The second moment of \( X \) is

\[
E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{0}^{1} \frac{2x^2+2}{3} \, dx = \frac{x^4}{6} + \frac{2x^3}{9} \bigg|_{0}^{1} = \frac{7}{18}
\]

The variance of \( X \) is \( \text{Var}[X] = E[X^2] - (E[X])^2 = 7/18 - (5/9)^2 = 13/162 \).
(b) The expected value of $Y$ is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) \, dy = \int_{0}^{2} y \frac{2y+1}{6} \, dy = \frac{y^2}{12} + \frac{y^3}{9} \bigg|_0 = \frac{11}{9}$$

The second moment of $Y$ is

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) \, dy = \int_{0}^{2} y^2 \frac{2y+1}{6} \, dy = \frac{y^3}{18} + \frac{y^4}{12} \bigg|_0 = \frac{16}{9}$$

The variance of $Y$ is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/81$.

(c) The correlation of $X$ and $Y$ is

$$E[XY] = \int \int xy f_{X,Y}(x,y) \, dx \, dy = \int_{0}^{1} \int_{0}^{2} xy \left( \frac{x+y}{3} \right) \, dy \, dx$$

$$= \int_{0}^{1} \left( \frac{x^2y^2}{6} + \frac{xy^3}{9} \bigg|_{y=0} \right) \, dx$$

$$= \int_{0}^{1} \left( \frac{2x^2}{3} + \frac{8x}{9} \right) \, dx = \frac{2x^3}{9} + \frac{4x^2}{9} \bigg|_0 = \frac{2}{3}$$


(d) The expected value of $X$ and $Y$ is


(e) By Theorem 5.10,

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X,Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162}$$

**Problem 5.5.2**

(a) The first moment of $X$ is

$$E[X] = \int_{0}^{1} \int_{0}^{1} 4x^2y \, dy \, dx = \int_{0}^{1} 2x^2 \, dx = \frac{2}{3}$$

The second moment of $X$ is

$$E[X^2] = \int_{0}^{1} \int_{0}^{1} 4x^3y \, dy \, dx = \int_{0}^{1} 2x^3 \, dx = \frac{1}{2}$$

The variance of $X$ is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/2 - (2/3)^2 = 1/18$. 


(b) The mean of $Y$ is

$$E[Y] = \int_0^1 \int_0^1 4xy^2 \, dy \, dx = \int_0^1 \frac{4x}{3} \, dx = \frac{2}{3}$$

The second moment of $Y$ is

$$E[Y^2] = \int_0^1 \int_0^1 4xy^3 \, dy \, dx = \int_0^1 x \, dx = \frac{1}{2}$$

The variance of $Y$ is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/2 - (2/3)^2 = 1/18$.

(c) To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^1 4x^2 y^2 \, dy \, dx = \int_0^1 \frac{4x^2}{3} \, dx = \frac{4}{9}$$

The covariance is thus

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{4}{9} - \left( \frac{2}{3} \right)^2 = 0$$

(d) $E[X + Y] = E[X] + E[Y] = \frac{2}{3} + \frac{2}{3}$

(e) By Theorem 5.10, the variance of $X + Y$ is

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X,Y] = 1/18 + 1/18 + 0 = 1/9$$

Problem 5.6.1

(a) Given the event $A = \{X + Y \leq 1\}$, we wish to find $f_{X,Y|A}(x,y)$. First we find

$$P[A] = \int_0^1 \int_0^{1-x} 6e^{-2x+3y} \, dy \, dx = 1 - 3e^{-2} + 2e^{-3}$$

So then

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}} & x + y \leq 10, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.6.2

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} \frac{(x+y)/3}{2} & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$
(a) The probability that $Y \leq 1$ is

$$P[A] = P[Y \leq 1] = \int \int_{y \leq 1} f_{X,Y}(x,y) \, dx \, dy$$

$$= \int_0^1 \int_0^{1} \frac{x+y}{3} \, dy \, dx$$

$$= \int_0^1 \left( \frac{xy}{3} + \frac{y^2}{6} \right)|_{y=0}^{y=1} \, dx$$

$$= \int_0^1 \frac{2x+1}{6} \, dx = \frac{x^2}{6} + \frac{x}{6}|_{0}^{1} = \frac{1}{3}$$

(b) By Definition 5.5, the conditional joint PDF of $X$ and $Y$ given $A$ is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y|A}(x,y) = \begin{cases} x+y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) From $f_{X,Y|A}(x,y)$, we find the conditional marginal PDF $f_{X|A}(x)$. For $0 \leq x \leq 1$,

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y) \, dy = \int_0^1 (x+y) \, dy = xy + \frac{y^2}{2}\bigg|_{y=0}^{y=1} = x + \frac{1}{2}$$

The complete expression is

$$f_{X|A}(x) = \begin{cases} x+1/2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) For $0 \leq y \leq 1$, the conditional marginal PDF of $Y$ is

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y) \, dx = \int_0^1 (x+y) \, dx = \frac{x^2}{2} + xy\bigg|_{x=0}^{x=1} = y + \frac{1}{2}$$

The complete expression is

$$f_{Y|A}(y) = \begin{cases} y+1/2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.7.1

$$f_{X,Y}(x,y) = \begin{cases} (x+y) & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) The conditional PDF $f_{X|Y}(x|y)$ is defined for all $y$ such that $0 \leq y \leq 1$.

(b) For $0 \leq y \leq 1$,

$$f_{X|Y}(x) = \frac{f_{X,Y}(x,y)}{f_{Y}(y)} = \frac{(x+y)}{f_{Y}(x+y)} \bigg|_{y=0}^{y=1} = \begin{cases} \frac{(x+y)}{y+1/2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
(c) $f_{Y|X}(y|x)$ is defined for all values of $x$ in the interval $[0, 1]$.

(d) For $0 \leq x \leq 1$,

$$f_{Y|X}(y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{(x+y)}{\int_0^1 (x+y) \, dx} = \begin{cases} \frac{(x+y)}{y+1/2} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

**Problem 5.8.1**

$X$ and $Y$ are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) To calculate $P[X > Y]$, we use the joint PDF $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

$$P[X > Y] = \int_{x>y} f_X(x)f_Y(y) \, dx \, dy = \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3} \, dx \, dy$$

$$= \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3} \, dx \, dy$$

$$= \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3} \, dx \, dy$$

$$= \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3} \, dx \, dy$$

$$= \int_0^\infty \frac{1}{2}e^{-1/2+1/3} \, dy = \frac{1/2}{1/2 + 2/3} = \frac{3}{7}$$

(b) Since $X$ and $Y$ are exponential random variables with parameters $\lambda_X = 1/3$ and $\lambda_Y = 1/2$, Appendix A tells us that $E[X] = 1/\lambda_X = 3$ and $E[Y] = 1/\lambda_Y = 2$. Since $X$ and $Y$ are independent, the correlation is $E[XY] = E[X]E[Y] = 6$.

(c) Since $X$ and $Y$ are independent, Cov $[X, Y] = 0$.  

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