Problem 4.1.2

\[ F_V(v) = \begin{cases} 
0 & v < -5 \\
c(v+5)^2 & -5 \leq v < 7 \\
1 & v \geq 7
\end{cases} \]

(a) For \( V \) to be a continuous random variable, \( F_V(v) \) must be a continuous function. This occurs if we choose \( c \) such that \( F_V(v) \) doesn’t have a discontinuity at \( v = 7 \). We meet this requirement if \( c(7+5)^2 = 1 \). This implies \( c = 1/144 \).

(b) \[ P[V > 4] = 1 - P[V \leq 4] = 1 - F_V(4) = 1 - 81/144 = 63/144 \]

(c) \[ P[-3 < V \leq 0] = F_V(0) - F_V(-3) = 25/144 - 4/144 = 21/144 \]

(d) Since \( 0 \leq F_V(v) \leq 1 \) and since \( F_V(v) \) is a nondecreasing function, it must be that \( -5 \leq a \leq 7 \). In this range,

\[ P[V > a] = 1 - F_V(a) = 1 - (a+5)^2/144 = 2/3 \]

The unique solution in the range \(-5 \leq a \leq 7 \) is \( a = 4\sqrt{3} - 5 = 1.928 \).

Problem 4.2.1

\[ f_X(x) = \begin{cases} 
 cx & 0 \leq x \leq 2 \\
0 & \text{otherwise}
\end{cases} \]

(a) From the above PDF we can determine the value of \( c \) by integrating the PDF and setting it equal to 1.

\[ \int_0^2 cx \, dx = 2c = 1 \]

Therefore \( c = 1/2 \).

(b) \[ P[0 \leq X \leq 1] = \int_0^1 x \, dx = 1/4 \]

(c) \[ P[-1/2 \leq X \leq 1/2] = \int_{-1/2}^{1/2} \frac{x}{2} \, dx = 1/16 \]

(d) The CDF of \( X \) is found by integrating the PDF from 0 to \( x \).

\[ F_X(x) = \int_0^x f_X(x') \, dx' = \begin{cases} 
0 & x < 0 \\
x^2/4 & 0 \leq x \leq 2 \\
1 & x > 2
\end{cases} \]
Problem 4.3.1

\[ f_X(x) = \begin{cases} 
1/4 & -1 \leq x \leq 3 \\
0 & \text{otherwise}
\end{cases} \]

We recognize that \( X \) is a uniform random variable from \([-1, 3]\).

(a) \( E[X] = 1 \) and \( \text{Var}[X] = \frac{(3+1)^2}{12} = 4/3 \).

(b) The new random variable \( Y \) is defined as \( Y = h(X) = X^2 \). Therefore

\[ h(E[X]) = h(1) = 1 \]

and

\[ E[h(X)] = E[X^2] = \text{Var}[X] + E[X]^2 = 4/3 + 1 = 7/3 \]

Finally

\[ E[Y] = E[h(X)] = E[X^2] = 7/3 \]
\[ \text{Var}[Y] = E[X^4] - E[X^2]^2 = \int_{-1}^{3} x^4 \, dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9} \]

Problem 4.3.4

(a) We can find the expected value of \( X \) by direct integration of the given PDF.

\[ f_Y(y) = \begin{cases} 
y/2 & 0 \leq y \leq 2 \\
0 & \text{otherwise}
\end{cases} \]

The expectation is

\[ E[Y] = \int_{0}^{2} \frac{y^2}{2} \, dy = 4/3 \]

(b)

\[ E[Y^2] = \int_{0}^{2} \frac{y^3}{2} \, dy = 2 \]
\[ \text{Var}[Y] = E[Y^2] - E[Y]^2 = 2 - (4/3)^2 = 2/9 \]