Problem 3.1.2
On the $X,Y$ plane, the joint PMF is

$$P_{X,Y}(x,y) = \begin{cases} 
c & (x=2, y=1) 
2c & (x=2, y=0) 
\frac{3}{2}c & (x=2, y=1) 
\frac{1}{2}c & (x=0, y=1) 
\end{cases}$$

(a) To find $c$, we sum the PMF over all possible values of $X$ and $Y$. We choose $c$ so the sum equals one.

$$\sum_{x} \sum_{y} P_{X,Y}(x,y) = \sum_{x=-2,0} \sum_{y=-1,0,1} c|x+y| = 6c + 2c + 6c = 14c$$

Thus $c = \frac{1}{14}$.

(b) $P[Y < X] = P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1)
= c + c + 2c + 3c = 7c = \frac{1}{2}$

(c) $P[Y > X] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1)
= 3c + 2c + c + c = 7c = \frac{1}{2}$

(d) From the sketch of $P_{X,Y}(x,y)$ given above, $P[X = Y] = 0$.

(e) $P[X < 1] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,-1) + P_{X,Y}(0,1)
= 8c = \frac{8}{14}$
Problem 3.3.2
On the $X, Y$ plane, the joint PMF is

\[ P_{X,Y}(x,y) \]

(a) To find $c$, we sum the PMF over all possible values of $X$ and $Y$. We choose $c$ so the sum equals one.

\[
\sum_x \sum_y P_{X,Y}(x,y) = \sum_{x=-2} \sum_{y=-1} c|x+y| = 6c + 2c + 6c = 14c
\]

Thus $c = 1/14$.

(b) \[ P[Y < X] = P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1) \]
\[ = c + c + 2c + 3c = 7c = 1/2 \]

(c) \[ P[Y > X] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1) \]
\[ = 3c + 2c + c + c = 7c = 1/2 \]

(d) From the sketch of $P_{X,Y}(x,y)$ given above, $P[X = Y] = 0$.

(e) \[ P[X < 1] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,-1) + P_{X,Y}(0,1) \]
\[ = 8c = 8/14 \]

Problem 3.4.2
In Problem 3.2.2, we found that the joint PMF of $X$ and $Y$ was
The expected values and variances were found to be
\[ E[X] = 0 \quad \text{Var}[X] = 24/7 \]
\[ E[Y] = 0 \quad \text{Var}[Y] = 5/7 \]

We will need these results in the solution to this problem.

(a) Random variable \( W = 2^{XY} \) has expected value
\[
E[2^{XY}] = \sum_{x=-2,0,2,y=-1,0,1} 2^{xy} P_{X,Y}(x,y)
= 2^{-2(-1)} \frac{3}{14} + 2^{-2(0)} \frac{2}{14} + 2^{-2(1)} \frac{1}{14} + 2^{0(-1)} \frac{1}{14} + 2^{0(1)} \frac{1}{14} \\
+ 2^{2(-1)} \frac{1}{14} + 2^{2(0)} \frac{2}{14} + 2^{2(1)} \frac{3}{14}
= 61/28
\]

(b) The correlation of \( X \) and \( Y \) is
\[
r_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}} = \frac{4}{7}
\]

(c) The covariance of \( X \) and \( Y \) is
\[
\sigma_{X,Y} = E[XY] - E[X]E[Y] = 4/7
\]

(d) The correlation coefficient is
\[
\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}} = \frac{2}{\sqrt{30}}
\]

**Problem 3.5.2**
The event \( B \) occurs iff \( X \leq 5 \) and \( Y \leq 5 \) and has probability
\[
P[B] = P[X \leq 5, Y \leq 5] = \sum_{x=1}^{5} \sum_{y=1}^{5} 0.01 = 0.25
\]

From Theorem 3.11,
\[
P_{X,Y|B}(x,y) = \begin{cases} \frac{P_{X,Y}(x,y)}{P[B]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.04 & x = 1, \ldots, 5; y = 1, \ldots, 5 \\ 0 & \text{otherwise} \end{cases}
\]
Problem 3.6.2
We can make a table of the possible outcomes and the corresponding values of $W$ and $Y$

<table>
<thead>
<tr>
<th>outcome</th>
<th>$P[\cdot]$</th>
<th>$W$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$hh$</td>
<td>$p^2$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$ht$</td>
<td>$p(1-p)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$th$</td>
<td>$p(1-p)$</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$tt$</td>
<td>$(1-p)^2$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the following table, we write the joint PMF $P_{W,Y}(w,y)$ along with the marginal PMFs $P_Y(y)$ and $P_W(w)$.

<table>
<thead>
<tr>
<th>$P_{W,Y}(w,y)$</th>
<th>$w = -1$</th>
<th>$w = 0$</th>
<th>$w = 1$</th>
<th>$P_Y(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 0$</td>
<td>0</td>
<td>$(1-p)^2$</td>
<td>0</td>
<td>$(1-p)^2$</td>
</tr>
<tr>
<td>$y = 1$</td>
<td>$p(1-p)$</td>
<td>0</td>
<td>$p(1-p)$</td>
<td>$2p(1-p)$</td>
</tr>
<tr>
<td>$y = 2$</td>
<td>0</td>
<td>$p^2$</td>
<td>0</td>
<td>$p^2$</td>
</tr>
<tr>
<td>$P_W(w)$</td>
<td>$p(1-p)$</td>
<td>$1-2p+2p^2$</td>
<td>$p(1-p)$</td>
<td></td>
</tr>
</tbody>
</table>

Using the definition $P_{W|Y}(w|y) = P_{W,Y}(w,y)/P_Y(y)$, we can find the conditional PMFs of $W$ given $Y$.

$P_{W|Y}(w|0) = \begin{cases} 1 & w = 0 \\ 0 & \text{otherwise} \end{cases}$

$P_{W|Y}(w|1) = \begin{cases} 1/2 & w = -1, 1 \\ 0 & \text{otherwise} \end{cases}$

Similarly, the conditional PMFs of $Y$ given $W$ are

$P_{Y|W}(y|1) = \begin{cases} 1 & y = -1 \\ 0 & \text{otherwise} \end{cases}$

$P_{Y|W}(y|0) = \begin{cases} (1-p)^2 & y = 0 \\ 1-2p+2p^2 & \text{otherwise} \end{cases}$

$P_{Y|W}(y|1) = \begin{cases} 1 & y = 1 \\ 0 & \text{otherwise} \end{cases}$

Problem 3.6.3
(a) First we observe that $A$ takes on the values $S_A = \{-1, 1\}$ while $B$ takes on values from $S_B = \{0, 1\}$. To construct a table describing $P_{A,B}(a,b)$ we build a table for all possible values of pairs $(A,B)$. The general form of the entries is

<table>
<thead>
<tr>
<th>$P_{A,B}(a,b)$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = -1$</td>
<td>$P_{B</td>
<td>A}(-1)$ $P_A(-1)$</td>
</tr>
<tr>
<td>$a = 1$</td>
<td>$P_{B</td>
<td>A}(1)$ $P_A(1)$</td>
</tr>
</tbody>
</table>

Now we fill in the entries using the conditional PMFs $P_{B|A}(b|a)$ and the marginal PMF $P_A(a)$. This yields

<table>
<thead>
<tr>
<th>$P_{A,B}(a,b)$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = -1$</td>
<td>$(1/3)(1/3)$</td>
<td>$(2/3)(1/3)$</td>
</tr>
<tr>
<td>$a = 1$</td>
<td>$(1/2)(2/3)$</td>
<td>$(1/2)(2/3)$</td>
</tr>
</tbody>
</table>
which simplifies to

<table>
<thead>
<tr>
<th>$P_{A,B}(a,b)$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = -1$</td>
<td>1/9</td>
<td>2/9</td>
</tr>
<tr>
<td>$a = 1$</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

(b) If $A = 1$, then the conditional expectation of $B$ is

$$E[B|A = 1] = \sum_{b=0}^{1} b P_{B|A}(b|1) = P_{B|A}(1|1) = 1/2$$

(c) Before finding the conditional PMF $P_{A|B}(a|1)$, we first sum the columns of the joint PMF table to find

$$P_{B}(b) = \begin{cases} 4/9 & b = 0 \\ 5/9 & b = 1 \end{cases}$$

The conditional PMF of $A$ given $B = 1$ is

$$P_{A|B}(a|1) = \frac{P_{A,B}(a,1)}{P_{B}(1)} = \begin{cases} 2/5 & a = -1 \\ 3/5 & a = 1 \end{cases}$$

(d) Now that we have the conditional PMF $P_{A|B}(a|1)$, calculating conditional expectations is easy.

$$E[A|B = 1] = \sum_{a=-1,1} a P_{A,B}(a|1) = -1(2/5) + (3/5) = 1/5$$

$$E[A^2|B = 1] = \sum_{a=-1,1} a^2 P_{A,B}(a|1) = 2/5 + 3/5 = 1$$

The conditional variance is then

$$\text{Var}[A|B = 1] = E[A^2|B = 1] - (E[A|B = 1])^2 = 1 - (1/5)^2 = 24/25$$

(e) To calculate the covariance, we need

$$E[A] = \sum_{a=-1,1} a P_{A}(a) = -1(1/3) + 1(2/3) = 1/3$$

$$E[B] = \sum_{b=0}^{1} b P_{B}(b) = 0(4/9) + 1(5/9) = 5/9$$

$$E[AB] = \sum_{a=-1,1} \sum_{b=0}^{1} ab P_{A,B}(a,b)$$

$$= -1(0)(1/9) - 1(1)(2/9) + 1(0)(1/3) + 1(1)(1/3) = 1/9$$

The covariance is just

Problem 3.7.3

(a) Normally, checking independence requires the marginal PMFs. However, in this problem, the zeroes in the table of the joint PMF $P_{X,Y}(x,y)$ allows us to verify very quickly that $X$ and $Y$ are dependent. In particular, $P_X(-1) = 1/4$ and $P_Y(1) = 14/48$ but

$$P_{X,Y}(-1,1) = 0 \neq P_X(-1)P_Y(1)$$

(b) To fill in the tree diagram, we need the marginal PMF $P_X(x)$ and the conditional PMFs $P_{Y|X}(y|x)$. By summing the rows on the table for the joint PMF, we obtain

<table>
<thead>
<tr>
<th>$P_{X,Y}(x,y)$</th>
<th>$y = -1$</th>
<th>$y = 0$</th>
<th>$y = 1$</th>
<th>$P_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = -1$</td>
<td>$3/16$</td>
<td>$1/16$</td>
<td>$0$</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$x = 0$</td>
<td>$1/6$</td>
<td>$1/6$</td>
<td>$1/6$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>$0$</td>
<td>$1/8$</td>
<td>$1/8$</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>

Now we use the conditional PMF definition $P_{Y|X}(y|x) = P_{X,Y}(x,y)/P_X(x)$ to write

$$P_{Y|X}(y|1) = \begin{cases} 3/4 & y = -1 \\ 1/4 & y = 0 \\ 0 & \text{otherwise} \end{cases} \quad P_{Y|X}(y|0) = \begin{cases} 1/3 & y = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Now we can use these probabilities to label the tree. The generic solution and the specific solution with the exact values are:

```
X = -1
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{Y</td>
<td>X}(-1,-1)</td>
</tr>
<tr>
<td>Y = -1</td>
<td>Y = 0</td>
</tr>
</tbody>
</table>

X = 0
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{Y</td>
<td>X}(-1,0)</td>
</tr>
<tr>
<td>Y = -1</td>
<td>Y = 0</td>
</tr>
</tbody>
</table>

X = 1
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{Y</td>
<td>X}(0,1)</td>
</tr>
<tr>
<td>P_{Y</td>
<td>X}(0,1)</td>
</tr>
<tr>
<td>P_{Y</td>
<td>X}(1,1)</td>
</tr>
</tbody>
</table>
```

```
X = -1
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3/4</td>
<td>1/4</td>
</tr>
<tr>
<td>Y = -1</td>
<td>Y = 0</td>
</tr>
</tbody>
</table>

X = 0
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>Y = -1</td>
<td>Y = 0</td>
</tr>
</tbody>
</table>

X = 1
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>Y = -1</td>
<td>Y = 0</td>
</tr>
</tbody>
</table>
```