Problem 2.4.1
Using the CDF given in the problem statement we find that
(a) \( P[Y < 1] = 0 \)
(b) \( P[Y \leq 1] = 1/4 \)
(c) \( P[Y > 2] = 1 - P[Y \leq 2] = 1 - 1/2 = 1/2 \)
(d) \( P[Y \geq 2] = 1 - P[Y < 2] = 1 - 1/4 = 3/4 \)
(e) \( P[Y = 1] = 1/4 \)
(f) \( P[Y = 3] = 1/2 \)
(g) From the staircase CDF of Problem 2.4.1, we see that \( Y \) is a discrete random variable. The
jumps in the CDF occur at the values that \( Y \) can take on. The height of each jump equals
the probability of that value. The PMF of \( Y \) is
\[
P_Y(y) = \begin{cases} 
1/4 & y = 1 \\
1/4 & y = 2 \\
1/2 & y = 3 \\
0 & \text{otherwise}
\end{cases}
\]

Problem 2.4.3
(a) Similar to the previous problem, the graph of the CDF is shown below.
\[
F_X(x) = \begin{cases} 
0 & x < -3 \\
0.4 & -3 \leq x < 5 \\
0.8 & 5 \leq x < 7 \\
1 & x \geq 7
\end{cases}
\]

(b) The corresponding PMF of \( X \) is
\[
P_X(x) = \begin{cases} 
0.4 & x = -3 \\
0.4 & x = 5 \\
0.2 & x = 7 \\
0 & \text{otherwise}
\end{cases}
\]
**Problem 2.5.3**

the PMF of $Y$ is

$$P_Y(y) = \begin{cases} 
1/4 & y = 1 \\
1/4 & y = 2 \\
1/2 & y = 3 \\
0 & \text{otherwise}
\end{cases}$$

The expected value of $Y$ is

$$E[Y] = \sum_y yP_Y(y) = 1(1/4) + 2(1/4) + 3(1/2) = 9/4$$

**Problem 2.5.5**

Problem 2.4.3, the PMF of $X$ is

$$P_X(x) = \begin{cases} 
0.4 & x = -3 \\
0.4 & x = 5 \\
0.2 & x = 7 \\
0 & \text{otherwise}
\end{cases}$$

The expected value of $X$ is

$$E[X] = \sum_x xP_X(x) = -3(0.4) + 5(0.4) + 7(0.2) = 2.2$$

**Problem 2.6.1**

the PMF of $Y$ is

$$P_Y(y) = \begin{cases} 
1/4 & y = 1 \\
1/4 & y = 2 \\
1/2 & y = 3 \\
0 & \text{otherwise}
\end{cases}$$

(a) Since $Y$ has range $S_Y = \{1, 2, 3\}$, the range of $U = Y^2$ is $S_U = \{1, 4, 9\}$. The PMF of $U$ can be found by observing that

$$P[U = u] = P[Y^2 = u] = P[Y = \sqrt{u}] + P[Y = -\sqrt{u}]$$

Since $Y$ is never negative, $P_U(u) = P_Y(\sqrt{u})$. Hence,

$$P_U(1) = P_Y(1) = 1/4 \quad P_U(4) = P_Y(2) = 1/4 \quad P_U(9) = P_Y(3) = 1/2$$

For all other values of $u$, $P_U(u) = 0$. The complete expression for the PMF of $U$ is

$$P_U(u) = \begin{cases} 
1/4 & u = 1 \\
1/4 & u = 4 \\
1/2 & u = 9 \\
0 & \text{otherwise}
\end{cases}$$
(b) From the PMF, it is straightforward to write down the CDF.

\[
F_U(u) = \begin{cases} 
0 & u < 1 \\
1/4 & 1 \leq u < 4 \\
1/2 & 4 \leq u < 9 \\
1 & u \geq 9 
\end{cases}
\]

(c) From Definition 2.14, the expected value of \( U \) is

\[
E[U] = \sum_u u P_U(u) = 1(1/4) + 4(1/4) + 9(1/2) = 5.75
\]

From Theorem 2.10, we can calculate the expected value of \( U \) as

\[
E[U] = E[Y^2] = \sum_y y^2 P_Y(y) = 1^2 (1/4) + 2^2 (1/4) + 3^2 (1/2) = 5.75
\]

As we expect, both methods yield the same answer.

**Problem 2.6.3**

Problem 2.4.3, the PMF of \( X \) is

\[
P_X(x) = \begin{cases} 
0.4 & x = -3 \\
0.4 & x = 5 \\
0.2 & x = 7 \\
0 & \text{otherwise}
\end{cases}
\]

(a) The PMF of \( W = -X \) satisfies

\[
P_W(w) = P[-X = w] = P_X(-w)
\]

This implies

\[
P_W(-7) = P_X(7) = 0.2 \quad P_W(-5) = P_X(5) = 0.4 \quad P_W(3) = P_X(-3) = 0.4
\]

The complete PMF for \( W \) is

\[
P_W(w) = \begin{cases} 
0.2 & w = -7 \\
0.4 & w = -5 \\
0.4 & w = 3 \\
0 & \text{otherwise}
\end{cases}
\]

(b) From the PMF, the CDF of \( W \) is

\[
F_W(w) = \begin{cases} 
0 & w < -7 \\
0.2 & -7 \leq w < -5 \\
0.6 & -5 \leq w < 3 \\
1 & w \geq 3
\end{cases}
\]

(c) From the PMF, \( W \) has expected value

\[
E[W] = \sum_w w P_W(w) = -7(0.2) + -5(0.4) + 3(0.4) = -2.2
\]
Problem 2.7.2
Whether a lottery ticket is a winner is a Bernoulli trial with a success probability of 0.001. If we buy one every day for 50 years for a total of $50 \cdot 365 = 18250$ tickets, then the number of winning tickets $T$ is a binomial random variable with mean

$$E[T] = 18250(0.001) = 18.25$$

Since each winning ticket grosses $1000, the revenue we collect over 50 years is $R = 1000T$ dollars. The expected revenue is

$$E[R] = 1000E[T] = 18250$$

But buying a lottery ticket everyday for 50 years, at $2.00 a pop isn’t cheap and will cost us a total of $18250 \cdot 2 = 36500$. Our net profit is then $Q = R - 36500$ and the result of our loyal 50 year patronage of the lottery system, is disappointing expected loss of

$$E[Q] = E[R] - 36500 = -18250$$

Problem 2.8.1
Given the following PMF

$$P_N(n) = \begin{cases} 0.2 & n = 0 \\ 0.7 & n = 1 \\ 0.1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) $E[N] = (0.2)0 + (0.7)1 + (0.1)2 = 0.9$

(b) $E[N^2] = (0.2)0^2 + (0.7)1^2 + (0.1)2^2 = 1.1$

(c) $\text{Var}[N] = E[N^2] - E[N]^2 = 1.1 - (0.9)^2 = 0.29$

(d) $\sigma_N = \sqrt{\text{Var}[N]} = \sqrt{0.29}$

Problem 2.8.7
For $Y = aX + b$, we wish to show that $\text{Var}[Y] = a^2 \text{Var}[X]$. We begin by noting that Theorem 2.12 says that $E[aX + b] = aE[X] + b$. Hence, by the definition of variance,

$$\text{Var}[Y] = E[(aX + b - (aE[X] + b))^2] = E[a^2(X - E[X])^2] = a^2E[(X - E[X])^2]$$

Since $E[(X - E[X])^2] = \text{Var}[X]$, the assertion is proved.

Problem 2.9.1
From the solution to Problem 2.4.1, the PMF of $Y$ is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases}$$
The probability of the event $B = \{Y < 3\}$ is $P[B] = 1 - P[Y = 3] = 1/2$. From Theorem 2.19, the conditional PMF of $Y$ given $B$ is

$$P_{Y|B}(y) = \begin{cases} \frac{P_{Y}(y)}{P[B]} & y \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2 & y = 1 \\ 1/2 & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

The conditional first and second moments of $Y$ are

$$E[Y|B] = \sum_{y} y P_{Y|B}(y) = 1(1/2) + 2(1/2) = 3/2$$

$$E[Y^2|B] = \sum_{y} y^2 P_{Y|B}(y) = 1^2(1/2) + 2^2(1/2) = 5/2$$

The conditional variance of $Y$ is

$$\text{Var}[Y|B] = E[Y^2|B] - (E[Y|B])^2 = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{5}{2} - \frac{9}{4} = \frac{1}{4}$$