CMPE 107
Homework 4

October 11, 2001

Problem Solutions: Yates and Goodman, 2.2.1 2.2.3 2.2.4 2.2.4 1.6.5 2.2.6 2.2.7 2.2.9 2.3.1
1.8.2 1.8.4 and 1.8.5
Contact TA with questions

Problem 2.2.1

(a) We wish to find the value of \( c \) that makes the PMF sum up to one.

\[
P_N(n) = \begin{cases} 
  c(1/2)^n & n = 0, 1, 2 \\
  0 & \text{otherwise}
\end{cases}
\]

Therefore, \( \sum_{n=0}^{2} P_N(n) = c + c/2 + c/4 = 1 \), implying \( c = 4/7 \).

(b) The probability that \( N \leq 1 \) is

\[
P[N \leq 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7
\]

Problem 2.2.3

(a) We must choose \( c \) to make the PMF of \( V \) sum to one.

\[
\sum_{v=1}^{4} P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1
\]

Hence \( c = 1/30 \).

(b) Let \( U = \{ u^2 | u = 1, 2, \ldots \} \) so that

\[
P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30}
\]

(c) The probability that \( V \) is even is

\[
P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3}
\]

(d) The probability that \( V > 2 \) is

\[
P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6}
\]
Problem 2.2.4

(a) We choose $c$ so that the PMF sums to one.

$$\sum_x P_X(x) = \frac{c}{2} + \frac{c}{4} + \frac{c}{8} = \frac{7c}{8} = 1$$

Thus $c = 8/7$.

(b) $P[X = 4] = P_X(4) = \frac{8}{7 \cdot 4} = \frac{2}{7}$

(c) $P[X < 4] = P_X(2) = \frac{8}{7 \cdot 2} = \frac{4}{7}$

(d) $P[3 \leq X \leq 9] = P_X(4) + P_X(8) = \frac{8}{7 \cdot 4} + \frac{8}{7 \cdot 8} = \frac{3}{7}$
Problem 1.6.5
For a sample space \( S = \{1, 2, 3, 4\} \) with equiprobable outcomes, consider the events
\[
A_1 = \{1, 2\} \quad A_2 = \{2, 3\} \quad A_3 = \{3, 1\}
\]
Each event \( A_i \) has probability \( 1/2 \). Moreover, each pair of events is independent since
\[
P[A_1A_2] = P[A_2A_3] = P[A_3A_1] = 1/4
\]
However, the three events \( A_1, A_2, A_3 \) are not independent since
\[
P[A_1A_2A_3] = 0 \neq P[A_1]P[A_2]P[A_3]
\]

Problem 2.2.6
The probability that a caller fails to get through in three tries is \((1 - p)^3\). To be sure that at least 95% of all callers get through, we need \((1 - p)^3 \leq 0.05\). This implies \( p = 0.6316 \).

Problem 2.2.7
In Problem 2.2.6, each caller is willing to make 3 attempts to get through. An attempt is a failure if all \( n \) operators are busy, which occurs with probability \( q = (0.8)^n \). Assuming call attempts are independent, a caller will suffer three failed attempts with probability \( q^3 = (0.8)^{3n} \). The problem statement requires that \((0.8)^{3n} \leq 0.05\). This implies \( n \geq 4.48 \) and so we need 5 operators.

Problem 2.2.9
(a) In the setup of a mobile call, the phone will send the “SETUP” message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability \( p \). Of course, the phone stops trying as soon as there is a success. Using \( r \) to denote a successful response, and \( n \) a non-response, the sample tree is

(b) We can write the PMF of \( K \), the number of “SETUP” messages sent as
\[
P_K(k) = \begin{cases} 
(1 - p)^{k-1}p & k = 1, 2, \ldots, 5 \\
(1 - p)^5p + (1 - p)^6 = (1 - p)^5 & k = 6 \\
0 & \text{otherwise}
\end{cases}
\]
Note that the expression for \( P_K(6) \) is different because \( K = 6 \) if either there was a success or a failure on the sixth attempt. In fact, \( K = 6 \) whenever there were failures on the first five attempts which is why \( P_K(6) \) simplifies to \((1 - p)^5\).

(c) Let \( B \) denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is \( P[B] = (1 - p)^6 \). To be sure that \( P[B] \leq 0.02 \), we need \( p \geq 1 - (0.02)^{1/6} = 0.479 \).
Problem 2.3.1

(a) If it is indeed true that \( Y \), the number of yellow M&M’s in a package, is uniformly distributed between 5 and 15, then the PMF of \( Y \), is

\[
P_Y(y) = \begin{cases} 
\frac{1}{11} & \text{if } y = 5, 6, 7, \ldots, 15 \\
0 & \text{otherwise}
\end{cases}
\]

(b) \[ P[Y < 10] = P_Y(5) + P_Y(6) + \cdots + P_Y(9) = \frac{5}{11} \]

(c) \[ P[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = \frac{3}{11} \]

(d) \[ P[8 \leq Y \leq 12] = P_Y(8) + P_Y(9) + \cdots + P_Y(12) = \frac{5}{11} \]

Problem 1.8.2

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is \( 4^3 = 64 \). If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of \( 4 \cdot 3 \cdot 2 = 24 \) possible codes.

Problem 1.8.4

In this case, the designated hitter must be chosen from the 15 field players. We can break down the experiment of choosing a starting lineup into two sub-experiments. The first is to choose 1 of the 10 pitchers, the second is to choose the remaining 9 batting positions out of the 15 field players. Here we are concerned about the ordering of our selections because a new ordering specifies a new starting lineup. So the total number of starting lineups when the DH is selected among the field players is

\[
\binom{10}{1} (15)_9
\]

Where \((15)_9\) is read as 15 ”permute” 9 and is equal to

\[(15)_9 = 15!/9! = 1,816,214,400\]

This gives a total of \( N_1 = 18,162,144,000 \) different lineups

Problem 1.8.5

When the DH can be chosen among all the players, including the pitchers, there are two cases:
1. The DH is a field player. In this case, the number of possible lineups, \( N_1 \), is given in Problem 1.8.4. In this case, the designated hitter must be chosen from the 15 field players. We can break down the experiment of choosing a starting lineup into two sub-experiments. The first is to choose 1 of the 10 pitchers, the second is to choose the remaining 9 batting positions out of the 15 field players. Here we are concerned about the ordering of our selections because a new ordering specifies a new starting lineup. So the total number of starting lineups when the DH is selected among the field players is

\[
\binom{10}{1} (15)_9
\]

where \((15)_9\) is

\[
(15)_9 = 15!/9! = 1,816,214,400
\]

This gives a total of \( N_1 = 18,162,144,000 \) different lineups.

2. The DH is a pitcher. In this case, there are 10 choices for the pitcher, 10 choices for the DH among the pitchers (including the pitcher batting for himself), \( \binom{15}{8} \) choices for the field players, and 9! ways of ordering the batters into a lineup. The number of possible lineups is

\[
N_2 = (10)(10) \binom{15}{8} 9! = 233,513,280,000
\]

The total number of ways of choosing a lineup is

\[
N_1 + N_2 = 251,675,424,000
\]