Theorem
Let \( F : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}} \) be a Boolean function. Then its dual \( F^d \) is given by the formula

\[
F^d(x_1, \ldots, x_n) = \overline{F(x_1, \ldots, x_n)}
\]

Remark: The Duality Principle follows immediately from this result since then \( F = G \) iff \( F^d = G^d \).

We leave the proof of the above Theorem as an exercise (Problem #27 D. 70%). Hint: Use the recursive definition of Boolean expression and induction on the expression which represents \( F \), the key step is to use the identities \( \overline{x \cdot y} = x + \overline{y} \) and \( \overline{x + y} = x \cdot \overline{y} \).

Example:
\[
F(x, y, z) = x \cdot \overline{y} + z \cdot \overline{x}
\]

\[
\overline{F(x, y, z)} = \overline{x \cdot \overline{y} + z \cdot \overline{x}} = (\overline{x \cdot \overline{y}}) \cdot (\overline{z \cdot \overline{x}}) = (x + \overline{y}) \cdot (z + \overline{x}) = F^d(x, y, z)
\]
10.2 Representing Boolean Functions

**DEFINITION**

A literal is a Boolean variable or its complement: \( x \) or \( \overline{x} \). A minterm of the variables \( x_1, x_2, \ldots, x_n \) is a product of the form:

\[ y_1 y_2 \ldots y_n \]

where each \( y_i \) is either \( x_i \) or \( \overline{x_i} \), i.e., a minterm is a product of literals.

**NOTE** that a minterm is true for exactly one combination of values of its variables.

**Example** \( x \overline{y} z \) is a minterm of \( x, y, z \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( \overline{y} )</th>
<th>( \overline{z} )</th>
<th>( x \overline{y} z )</th>
</tr>
</thead>
<tbody>
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</table>
Observe that every Boolean function can be represented as a sum of minterms of its variables. This expression is called the sum of products representation or disjunctive normal form (DNF) of the function.

Ex. \( F(x, y, z) = \bar{x}(y + \bar{z}) \)

\[
= \bar{x}y + \bar{x}\bar{z} \\
= \bar{x} \cdot 1 + \bar{x} \cdot 1 \cdot \bar{z} \\
= \bar{x}y \bar{z} + \bar{x}(y + \bar{y}) \bar{z} \\
= \bar{x}y \bar{z} + \bar{x}y' \bar{z} + \bar{x}y \bar{z} + \bar{x}y \bar{z} \\
= \bar{x}y' \bar{z} + \bar{x}y \bar{z} + \bar{x}y \bar{z}
\]

To find the DNF of a function, given its truth table, write the minterm corresponding to each 1 in the output column, then take the sum of these minterms.

Ex.\[
\begin{array}{ccc|c}
 x & y & z & g(x,y,z) \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 \rightarrow \bar{x}y \bar{z} \\
 0 & 1 & 0 & 0 \\
 0 & 1 & 1 & 0 \\
 1 & 0 & 0 & 0 \\
 1 & 0 & 1 & 1 \rightarrow x'z \\
 1 & 1 & 0 & 1 \rightarrow x'z \\
 1 & 1 & 1 & 0 \\
\end{array}
\]

\( \therefore g(x,y,z) = \bar{x}y'z + x'yz + xyz \)
This procedure shows that any Boolean function can be placed in DNF, and in particular, the mapping

\[ \{ \text{expressions} \} \rightarrow \{ \text{functions} \} \]

is onto. Note also that the DNF of a function is unique up to order of terms.

Likewise, any Boolean function can be represented by a product of some expression called the conjunctive normal form (CNF).

This form can be obtained by taking De Morgan's law twice and using \( F \cup F = F \).

Ex. \( F(x,y) = xy + x\bar{y} \)

\[
(x\bar{y})_d = (x+y)(\bar{x}+\bar{y})
\]
\[
= xy + \bar{x}\bar{y} + xy + y\bar{y}
\]
\[
= 0 + xy + xy + 0
\]
\[
= xy + x\bar{y}
\]

\[
(xy + x\bar{y})_d = (xyz + x\bar{y})_d
\]
\[
= (x+y)(x+\bar{y})
\]

\[
\therefore F(x,y) = (x+y)(x+\bar{y})
\]
Since every boolean function can be represented using the operators $\cdot$, $+$, $-$, we say that the set 

$$\{0, +, -\}$$

is functionally complete.

Actually this is not the smallest such set. Using the identity $x+y = \overline{\overline{x} \cdot \overline{y}}$ we can eliminate all occurrences of $+$, showing that $\{0, -\}$ is also functionally complete.

Likewise $x \cdot y = \overline{x} + \overline{y}$ shows that $\{+, -\}$ is functionally complete as well.

It is also true that there are complete sets consisting of just one operator.

\[\begin{array}{c|c|c|c}
  x & y & x \mid y & x \downarrow y \\
  \hline
  0 & 0 & 1 & 1 \\
  0 & 1 & 1 & 0 \\
  1 & 0 & 1 & 0 \\
  1 & 1 & 0 & 0 \\
\end{array}\]
Observe that \( \{1\} \) is functionally complete since

\[
\overline{x} = x \cdot x = \frac{x}{x} \quad xy = (x \cdot y) \mid (x \cdot y)
\]

and the fact that \( \{0, 1\} \) is functionally complete.

Likewise, \( \{ \uparrow \} \) is complete since

\[
\overline{x} = x \downarrow x = \frac{x}{x} \quad x + y = (x \downarrow y) \uparrow (x \downarrow y)
\]

and the fact that \( \{ +, \uparrow \} \) is complete.

Exercise: Prove all of the above identities.

Exercise:
Find the DNF of \( x \cdot y \) and \( x \downarrow y \), and show that

\[
(x \downarrow y)^f = x \cdot y
\]