linear 2nd order homogeneous rec. relation with coeff. coeff.

\[ x_n = c_1 x_{n-1} + c_2 x_{n-2} \quad (n \geq 0) \]

Then (superposition principle)

if \((a_n)\) and \((b_n)\) are solutions to \((1)\), then so is \((\alpha a_n + \beta b_n)\)

for any \(\alpha, \beta \in \mathbb{R}\).

Proof.

Let \(x_n = \alpha a_n + \beta b_n\). Then
\[ 2L_2 \Delta x = c_1 x_{n-1} + c_2 x_{n-2} \]

\[ = c_1 (\alpha a_{n-1} + \beta b_{n-1}) + c_2 (\alpha a_{n-2} + \beta b_{n-2}) \]

\[ = \alpha (c_1 a_{n-1} + c_2 a_{n-2}) + \beta (c_1 b_{n-1} + c_2 b_{n-2}) \]

\[ = \alpha \Delta x_n + \beta \Delta b_n \]

\[ = \Delta x_n = L \Delta \xi \]

To solve (1), guess a solution of the form \( x_n = r^n \). Substitute into (1) to obtain

\[ r^n = c_1 r^{n-1} + c_2 r^{n-2} \]

divide both sides by \( r^{n-2} \).
(2) \[ r^2 = c_1 r + c_2 \]

or

(3) \[ r^2 - c_1 r - c_2 = 0 \]

(2) or (3) is called the characteristic equation for (1).

**Thm**

If \( r_0 \) is any root of the characteristic eqn (2) or (3), then \( x_n = r_0^n \) is a solution to (1).

**Proof**: exercise or see book.
The key to finding solution to (11) is finding roots of quadratic (3):

\[ r_1 = \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}, \quad r_2 = \frac{c_1 - \sqrt{c_1^2 + 4c_2}}{2} \]

There are 3 cases:

(i) \( r_1 \neq r_2 \) are distinct real \( (c_1^2 + 4c_2 > 0) \)

(ii) \( r_1 = r_2 \) repeated real roots \( (c_1^2 + 4c_2 = 0) \)

(iii) \( r_1 \neq r_2 \) complex conjugate roots \( (c_1^2 + 4c_2 < 0) \)

We consider case (i) only. (see text for other cases.)
Ex: \[ a_n = 5a_{n-1} - 6a_{n-2} \]
\[ a_0 = 1 \]
\[ a_1 = 0 \]

Char eqn: \( r^2 - 5r + 6 = 0 \)

\((r-2)(r-3) = 0\)

\( r_1 = 2, r_2 = 3 \)

Thm: If \( r_1, r_2 \) are distinct real roots of \( (2) \), then every solution to \((1)\) is of the form

\[ x_n = \alpha r_1^n + \beta r_2^n \]

Proof: See text.
Returning to example...

\[ a_n = \alpha 2^n + \beta 3^n \]

to find \( \alpha, \beta \) use init. cond.

\[
\begin{align*}
\alpha 2^0 + \beta 3^0 &= 1 \\
\alpha 2^1 + \beta 3^1 &= 0
\end{align*}
\]

\[
\begin{align*}
\alpha + \beta &= 1 \\
2\alpha + 3\beta &= 0 \\
2\alpha + 2\beta &= 2
\end{align*}
\]

\[ \beta = -2 \Rightarrow \alpha = 3 \]

\[
\therefore a_n = 3 \cdot 2^n - 2 \cdot 3^n
\]
Ex. \( x_n = 4x_{n-2} \), \( x_0 = 0 \), \( x_1 = 4 \)

** Chew eqn: \( r^2 = 4 \)**

\[
ger^2 - 4 = 0
\]

\[
(r - 2)(r + 2) = 0
\]

\[
r_1 = 2, \quad r_2 = -2
\]

\[
\therefore \quad x_n = \alpha 2^n + \beta (-2)^n
\]

\[
\begin{cases}
\alpha + \beta = 0 \\
-2\alpha - 2\beta = 4
\end{cases} \quad \Rightarrow \begin{cases}
\alpha + \beta = 0 \\
2\alpha = 2 \quad \Rightarrow \alpha = 1
\end{cases}
\]

\[
\therefore \quad \beta = -1
\]

\[
\therefore \quad x_n = 2^n + (-1)(-2)^n
\]

\[
\boxed{x_n = 2^n - (-2)^n
\]

\[
x_n = 2^n - (-1)^n 2^n
\]
\[ x_n = 2^n \left( 1 - (-1)^n \right) \]

\[
\begin{align*}
x_n = & \begin{cases} 
0 & \text{n even} \\
\frac{n+1}{2} & \text{n odd}
\end{cases}
\end{align*}
\]

Ex. (Fibonacci)

\[
\begin{cases}
F_n = F_{n-1} + F_{n-2} & \text{n \geq 2} \\
F_0 = 0 \\
F_1 = 1
\end{cases}
\]

Char. eqn.: \[ r^2 = r + 1 \]

\[ r^2 - r - 1 = 0 \]

\[ r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2} \]

\[ \phi \]

Golden Ratio.
\[ F_n = \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[
\begin{align*}
\alpha + \beta &= 0 \\
\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) &= 1 \\
\sqrt{5} (\alpha + \beta) + \sqrt{5} (\alpha - \beta) &= 2
\end{align*}
\]

\[
\begin{align*}
\alpha + \beta &= 0 \quad \Rightarrow \quad \beta = -\alpha \\
\alpha - \beta &= \frac{2}{\sqrt{5}} \\
\end{align*}
\]

\[ 2\alpha = \frac{2}{\sqrt{5}} \quad \Rightarrow \quad \alpha = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad \beta = -\frac{1}{\sqrt{5}} \]

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[ F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \]
since \( \left( \frac{1 - \sqrt{5}}{2} \right)^n \to 0 \) as \( n \to \infty \),

we have

\[ F_n = \text{nearest int to} \quad \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \]

for sufficiently large \( n \).
Ex.

In how many ways can a 2xn rectangular walkway be tiled using only 1x1 and 2x2 tiles?

Walk: \[ \begin{array}{cccc|c}
1 & 1 & 1 & 1 & \vdots \\
\hline
1 & 1 & 1 & 1 & \ddots \\
\hline
1 & 1 & 1 & 1 & 1 \\
\end{array} \]

Tiles: \[ \begin{array}{c}
1 \times 2 \\
2 \times 2 \\
\end{array} \]

There are 3 mutually exclusive ways such a walk can end.

Let \( A_n \) = \# of such tilings.
1.) 
\[
\text{An-1 ways}
\]

\[
\text{n-1}
\]

2.) 
\[
\text{An-2}
\]

\[
\text{n-2}
\]

3.) 
\[
\text{An-2}
\]

\[
\text{n-2}
\]

By the sum rule:

\[
A_n = A_{n-1} + 2A_{n-2}
\]

\[
\begin{cases}
A_0 = 1 \\
A_1 = 1 \\
A_2 = 3
\end{cases}
\]
char eqn. : \( r^2 = 1 + 2 \)

\[ \therefore r^2 - r - 2 = 0 \]

\[ (r + 1)(r - 2) = 0 \]

\[ r_1 = -1, \quad r_2 = 2 \]

\[ A_n = \alpha (-1)^n + \beta 2^n \]

\[
\begin{align*}
\alpha + \beta &= 1 \\
-\alpha + 2\beta &= 1
\end{align*}
\]

\[ \Rightarrow \alpha = 1 - \beta = 1 - \frac{2}{3} = \frac{1}{3} \]

\[ 3\beta = 2 \Rightarrow \beta = \frac{2}{3} \]

\[ \therefore A_n = \frac{1}{3} (-1)^n + \frac{2}{3} (2^n) \]

\[ A_n = \frac{1}{3} \left( 2^{n+1} + (-1)^n \right) \]
9.1 Relations

Note: the word "relation" is used here in a different sense than in chapter 8.

Let $A, B$ be sets.

Define

A binary relation from $A$ to $B$ in a subset: $R \subseteq A \times B$.

Notation

If $(x, y) \in R$ we say $x$ is related to $y$ by $R$ and write $xRy$

i.e. $xRy$ iff $(x, y) \in R$. 
Also write $x R y$ to mean $(x, y) \in R$.

Example: $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$ let $R = \{(1, x), (1, z), (2, y), (2, z), (3, x), (4, y)\}$ is a relation from $A$ to $B$.

So $2 R y$ but $2 \not{R} x$.

\[\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & \rightarrow & 3 & \rightarrow \\
2 & \rightarrow & 4 & \rightarrow \\
3 & \rightarrow & 1 & \rightarrow \\
4 & \rightarrow & 2 & \rightarrow \\
\end{array}\]
Recall if \( A, B \) are finite then 
\[ |A \times B| = |A| \cdot |B| \]. In this case, the set of relations from \( A \) to \( B \) is finite: \( \mathcal{P}(A \times B) \)

\[ |\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{\frac{|A| \cdot |B|}{2}} \]

Note: \( \uparrow \quad \uparrow \)
- \( \downarrow \quad \downarrow \)
- domain codomain

**Definition:** If \( f : A \rightarrow B \), the graph \( \text{graph}(f) \) is

\[ \left\{ (x, f(x)) \mid x \in A \right\} \subseteq A \times B \]
Note: \( R \subseteq A \times A \) in the graph of a function \( f: A \rightarrow \mathbb{R} \) if:

for all \( x \in A \), there exists a unique \( y \in B \) such that \( xRy \).

In this case write: \( y = f(x) \).

**Def.**

A relation on \( A \) is a relation from \( A \) to \( A \): \( R \subseteq A \times A \).

**Ex.** \( R \subseteq \mathbb{Z} \times \mathbb{Z} \)

\[ R = \{ (n, m) \mid n < m \} \]

i.e. "\( R \) equals "\(<""
i.e. \( n \equiv m \pmod{n} \), \( n < m \)

This is the "less-than" relation.

Other relations on \( \mathbb{Z} \):

\[
\begin{align*}
< & \quad \text{less than} \\
\leq & \quad \text{less than or eq.} \\
> & \quad \text{greater than} \\
\geq & \quad \text{greater than or eq.} \\
\equiv & \quad \text{eq.} \\
\equiv_m & \quad \text{congruence mod m}
\end{align*}
\]
1st $\mathbb{Z}^* = \mathbb{Z} - \{0\}$

Relation on $\mathbb{Z}^*$: $\mid$ divisibility

$\sum (n, m) \mid n \mid m$ "n divides m" $\in \mathbb{Z}^* \times \mathbb{Z}^*$

How many relations on a finite set $A$?

$|\mathcal{P}(A \times A)| = 2^{4 \times 4|} = 2^{16}$

$\exists x \ A = \{0, 1\}$. There are

$2^2 = 2^4 = 16$

relations on $A$

Exercise: Write them all.
Properties of Relations

**Def:** \( R \subseteq A \times A \) is called reflexive iff for all \( x \in A \):

\[ xRx \]

**Ex.** Some reflexive relations on \( \mathbb{Z} \)

\[ =, \equiv_m, \leq, \geq \]

Proof: let \( x \in \mathbb{Z} \). must show \( x \equiv_m x \).

i.e. must show \( m \mid (x-x) \), i.e. \( m \mid 0 \).

This is true since \( 0 = 0 \cdot m \).
Ex. on $\mathbb{Z}^*$: is $\mid$ reflexive?

i.e. is $x \mid x$ for all $x \in \mathbb{Z}^*$.

True since $x = 1 \cdot x$. 