**Defn**

The range of \( f : A \to B \) is the set:

\[
\text{Range}(f) = \{ y \in B \mid \exists x \in A : y = f(x) \}
\]

\[
= \{ f(x) \mid x \in A \} \subseteq B
\]

We sometimes call \( f(x) \in B \) the image of \( x \) under \( f \). So

\[
\text{Range}(f) = \{ \text{all images under } f \}.
\]
Existence of $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

$\text{Range}(f) = \{0, 1, 4, 9, 16, 25, \ldots \}$

Existence of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$\text{Range}(f) = [0, \infty) = \{ y \in \mathbb{R} \mid y \geq 0 \}$
Let $f : A \rightarrow B$, $S \subseteq A$. The image of $S$ under $f$ is the set

$$f(S) = \{ f(x) \mid x \in S \} \subseteq \text{Range}(f) \subseteq B$$

Let $f : A \rightarrow B$, $T \subseteq B$. The preimage of $T$ under $f$ is the set

$$f^{-1}(T) = \{ x \in A \mid f(x) \in T \} \subseteq A$$
Ex 1: \( f: \mathbb{R} \to \mathbb{R}, \ f(x) = x^2 \)

\[ S = [2, 3] = \{ x \mid 2 \leq x \leq 3 \} \]

\[ \mathcal{F}(S) = [4, 9] = \{ y \mid 4 \leq y \leq 9 \} \]

\[ T = [5, 6] = \{ y \mid 5 \leq y \leq 6 \} \]

\[ f^{-1}(T) = [-\sqrt{6}, -\sqrt{5}] \cup [\sqrt{5}, \sqrt{6}] \]
A function $f: \mathbb{A} \to \mathbb{B}$ is called one-to-one, or injective, if

$$\forall x_1, x_2 \in \mathbb{A} : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

equivalently

$$\forall x_1, x_2 \in \mathbb{A} : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

For $f: \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$ is not injective: $f(-1) = f(1)$ but $-1 \neq 1$ so

$$\exists x_1, x_2 : f(x_1) = f(x_2) \text{ and } x_1 \neq x_2$$
Ex. \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( g(x) = 2x + 3 \) is injective.

Pick \( x_1, x_2 \in \mathbb{R} \) arbitrarily. Assume \( g(x_1) = g(x_2) \). Then

\[
2x_1 + 3 = 2x_2 + 3
\]

\[
\therefore \ 2x_1 = 2x_2
\]

\[
\therefore \ x_1 = x_2
\]

Define we say \( f : \mathbb{R} \rightarrow \mathbb{R} \) is strictly increasing iff \( x < y \Rightarrow f(x) < f(y) \).

Also \( f \) is strictly decreasing iff \( x < y \Rightarrow f(x) > f(y) \).
Thus a function that is strictly \text{inc} or \text{dec.} is necessarily \text{injective.}

\text{Definition}

\( f : A \rightarrow B \) is called \underline{onto} or \underline{surjective} \iff \text{Range}(f) = B, \ i.e.

\[ \forall y \in B, \exists x \in A : y = f(x) \]

\[ \begin{array}{c}
\text{onto} \\
\begin{array}{c}
A \\
\downarrow \\
\downarrow \\
\downarrow \\
B
\end{array}
\end{array} \quad \begin{array}{c}
\text{not onto} \\
\begin{array}{c}
A \\
\downarrow \\
\downarrow \\
\downarrow \\
B
\end{array}
\end{array} \]
Ex \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( f(x) = x^2 \) is not onto, since

\[ \exists y \in \mathbb{R}, \forall x \in \mathbb{R} : y \neq x^2 \]

Proof: let \( y = -1 \). \( \forall x \in \mathbb{R} : x^2 \neq -1 \).

Ex \( g : \mathbb{R} \rightarrow \mathbb{R} \), \( g(x) = 2x + 3 \) is onto, since:

let \( y \in \mathbb{R} \) be arbitrary. Must show there exists \( x \) s.t. \( y = 2x + 3 \).

let \( x = \frac{y-3}{2} \). Then

\[ g(x) = 2 \left( \frac{y-3}{2} \right) + 3 = (y-3) + 3 = y \]
**Definition**

A function \( f : A \rightarrow B \) is called a one-to-one correspondence or bijective if it is both injective and surjective.

**Example**

\[ g(x) = 2x + 3 \] is bijective.
Definition

The composition of $f: A \to B$ with $g: B \to C$ is the function $g \circ f: A \to C$ defined by

$$g \circ f(x) = g(f(x)) \quad \forall x \in A.$$ 

Example

Let $A = B = C = \mathbb{R}$, $f(x) = x^2$, $g(x) = 2x + 3$.

$$g \circ f(x) = g(f(x)) = g(x^2) = 2x^2 + 3$$

$$f \circ g(x) = f(g(x)) = f(2x + 3) = (2x + 3)^2$$

Note: $g \circ f \neq f \circ g$. 
Invertibility

Let $A, B$ be sets. Define

$$i_A : A \to A \text{ by } i_A(x) = x$$

$$i_B : B \to B \text{ by } i_B(y) = y$$

Observe that if $f : A \to B$

$$f \circ i_A = f = i_B \circ f$$

$i_A, i_B$ are called identity functions on $A$ and $B$ respectively.
Definition

$f : A \rightarrow B$ is invertible if there exists $g : B \rightarrow A$ such that

$$f \circ g = i_B \quad \text{and} \quad g \circ f = i_A.$$ 

$g$ is called an inverse function for $f$.

Thus

If $f$ is invertible then its inverse $g$ is unique.
Proof

Suppose there exist two functions $g_1 : B \to A$ and $g_2 : B \to A$ satisfying

$$f \circ g_1 = i_B, \quad g_1 \circ f = i_A$$
and $$f \circ g_2 = i_B, \quad g_2 \circ f = i_A.$$

Then

$$g_1 = i_A \circ g_1$$

$$= (g_2 \circ f) \circ g_1$$

$$= g_2 \circ (f \circ g_1)$$

$$= g_2 \circ i_B$$

$$= g_2.$$

Exercise:

Show $o$ is associative.
We may now speak at the inverse of \( f \), and write

\[ f^{-1} : B \to A \]

for \( g \).

**Theorem:**

\[ f : A \to B \text{ is invertible } \iff \]

\( f \) is bijective.

**Proof**

\((\Rightarrow)\) Suppose \( f \) is invertible, so

\[ f^{-1} : B \to A \text{ exists.} \]
So \( f \) is injective: \( \forall x_1, x_2 \in A : f(x_1) = f(x_2) \implies x_1 = x_2 \).

Let \( x_1, x_2 \in A \) be arbitrary. Assume that \( f(x_1) = f(x_2) \).

\[
\therefore \quad f^{-1}(f(x_1)) = f^{-1}(f(x_2))
\]

\[
\therefore \quad f \circ f(x) = f \circ f(x_2)
\]

\[
\therefore \quad i_A(x_1) = i_A(x_2)
\]

\[
\therefore \quad x_1 = x_2
\]

\( \therefore \) \( f \) is surjective:

\( \forall y \in B \exists x \in A : y = f(x) \)

Let \( y \in B \) be arbitrary. Set \( x = f^{-1}(y) \).

\[
\therefore \quad f(x) = f(f^{-1}(y)) = f \circ f^{-1}(y) = i_B(y) = y.
\]

\( \therefore \) \( f \) is bijective.
\((\Leftarrow)\) Suppose \(f\) is bijective.

Define \(g : B \rightarrow A\) by:

for each \(y \in B\)

\[ g(y) = \text{the unique } x \in A \text{ such that } f(x) = y. \]

Note: Such an \(x\) exists since \(f\) is surjective, and \(x\) is unique since \(f\) is injective.

Then necessarily

\[ f(g(y)) = y \text{ for all } y \in B \]

and

\[ g(f(x)) = x \text{ for all } x \in A \]
so \( f \circ g = i_B \) and \( g \circ f = i_A \)

so \( g = f^{-1} \) and \( f \) is invertible

\[ \text{Corollary} \]

\( f \) is invertible, so is \( f^{-1} \) and \( (f^{-1})^{-1} = f \).

\[ \text{Proof:} \]

follows from

\( f \circ f^{-1} = i_B \) and \( f^{-1} \circ f = i_A \).
Corollary

If $f$ is bijective, so is $f^{-1}$.

Ex. $g: \mathbb{R} \rightarrow \mathbb{R}, \ g(x) = 2x + 3$

is bijective, and is invertible, and

$$g^{-1}(x) = \frac{x - 3}{2}$$

Ex. $f: \mathbb{R} \rightarrow \mathbb{R}, \ f(x) = x^2$

is not bijective, so not invertible.

Ex. $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+, \ f(x) = x^2$

is bijective, so is invertible:

$$f^{-1}(x) = \sqrt{x}.$$