6.3 Permutations \& Combinations

**Notation:**

\[ P(n, k) = \text{# of } k\text{-Permutations of an } n\text{-set, } (0 \leq k \leq n) \]

\[ P(n, k) = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!} \]

**Special Case:**

\[ P(n, n) = n(n-1) \cdots 3 \cdot 2 \cdot 1 = n! \]

\[ \Rightarrow \text{# Permutations of an } n\text{-set} \]
Define a \( k \)-combination of a set \( S \) is an unordered arrangement of \( k \) elements of \( S \), i.e. a \( k \)-element subset of \( S \). \( 0 \leq k \leq |S| \)

Example: \( S = \{1, 2, 3, 4\} \), \( k = 0, 1, 2, 3 \)

\( k = 0 \): \( \emptyset \)

\( k = 1 \): \( \{1, 2, 3, 4\}, \{2, 3, 4\} \)

\( k = 2 \): \( \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3\} \)

\( k = 3 \): \( \{1, 2, 3\} \)

Note: \( \binom{n}{k} \) = \# of \( k \)-combinations of an \( n \)-set.
**other notation**

\[ C(n, k) = \binom{n}{k} = \binom{n}{k}_k = C_{n,k} = \binom{n}{k}_k = \binom{n}{k} = \binom{n}{k}_k \]

"n choose k"

Also called Binomial Coefficients

Thus let \(0 \leq k \leq n\). Then

\[ C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

**Proof.** Let \(S\) be a set with \(|S| = n\). A \(k\)-permutation of \(S\) can be constructed by performing the following tasks in succession:

\[ \text{...} \]
Choose a \( k \)-subset of \( S \): \( \binom{n}{k} \)

Choose a permutation of \( \{k, k\} \) those \( k \) elements.

By product rule

\[
\binom{n}{k} = \binom{n}{k} \cdot \binom{k}{k}
\]

\[
\therefore \frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!
\]

\[
\therefore \binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Ex: how many bit strings of length 10 contain exactly 6 1's?

\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\text{Pos: 1 2 3 4 5 6 7 8 9 10}
\end{array}
\]

\[\text{(# of such bit strings)} = \binom{10}{6} \text{ (# of ways of choosing which 6 of 10 positions are to be occupied by 1's).}\]

\[\binom{10}{6} = \frac{\binom{10}{6}}{6!} = \frac{10!}{6! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{4! \cdot 6!} = \frac{210}{210} = 210\]
could also count # of pos. to be occupied by 0's.

\[ C(10, 4) = \frac{10!}{4! \cdot 6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210 \]

In general:

\[
\left( \# \text{ bit strings of length } n \right) \\
\left( \text{ containing exactly } k \text{ 1's} \right) \\
= C(n, k) = \binom{n}{k}
\]

**Theorem**

\[
\binom{n}{k} = \binom{n}{n-k}
\]
Proof: (algebraic)
\[
\binom{n}{n-k} = \frac{n!}{(n-k)! (n-(n-k))!} = \frac{n!}{(n-k)! (n-k+k)!} = \frac{n!}{k! (n-k)!} = \binom{n}{k}.
\]

Categories of Proof:
- Algebraic: manipulate known identities.
- Combinatorial: argue that set $\frac{1}{n}$ Riff count same set.
Another proof: (combinatorial)

Proof:
Let \(|S| = n\). Define for \(0 \leq k \leq n\)

\[ S^{(k)} = \{ k \text{-subsets of } S \} \subseteq \mathcal{P}(S) \]

Define \(f : S^{(k)} \to S^{(n-k)}\)

\[ f(A) = \overline{A} = S - A \]

for \(A \in S^{(k)}\). Observe

\[ f(\overline{A}) = f(A) = \overline{A} = A \]

\[ \therefore f \text{ is invertible with } f^{-1} = f \]

\[ \therefore f \text{ is a bijection.} \]

\[ |S^{(k)}| = |S^{(n-k)}| \]

\[ \binom{n}{k} = \binom{n}{n-k} \].
A ternary string is a string on alphabet \{0, 1, 2\}.

Ex: find # of ternary strings of length \(n\) having \(k_1\) 0's and \(k_2\) 1's, where \(k_1, k_2 \in \mathbb{N}\) satisfy \(k_1 + k_2 \leq n\).

and \(= \binom{n}{k_1} \cdot \binom{n-k_1}{k_2}

= \frac{n!}{k_1! (n-k_1)!} \cdot \frac{(n-k_1)!}{k_2! (n-k_1-k_2)!}

= \frac{n!}{k_1! k_2! (n-k_1-k_2)!}
6.4 Binomial Coefficients

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\text{Observe:}

\[
\begin{align*}
\binom{0}{0} &= 1 \\
\binom{1}{0} &= 1, \quad \binom{1}{1} = 1 \\
\binom{2}{0} &= 1, \quad \binom{2}{1} = 2, \quad \binom{2}{2} = 1 \\
\binom{3}{0} &= 1, \quad \binom{3}{1} = 3, \quad \binom{3}{2} = 3, \quad \binom{3}{3} = 1
\end{align*}
\]
Pascal's Triangle.

\[\begin{array}{cccccc}
1 \\
0 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\vdots \\
\end{array}\]

Thus: (Pascal's identity)

Let \(1 \leq k \leq n\), then

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]
1st Root: (algebraic)

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)! (n-(k-1))!} + \frac{n!}{k! (n-k)!}
\]

\[
= \frac{n!}{(k-1)! (n-k+1)!} + \frac{n!}{k! (n-k)!}
\]

\[
= \frac{n! \cdot k}{k! (n-k+1)!} + \frac{n! \cdot (n-k+1)}{k! (n-k)!}
\]

\[
= \frac{n! \left( k + (n-k+1) \right)}{k! (n+1-k)!}
\]

\[
= \frac{n! \cdot (n+1)}{k! (n+1-k)!} = \frac{(n+1)!}{k! \left( (n+1)-k \right)!}
\]

\[
= \binom{n+1}{k}
\]
2nd Proof: (combinatorial)

Let $|T| = n + 1$, $x \in T$ and $S = T - \{x\}$.

Then $|S| = n$.

Let $S = \# \text{ of } k\text{-subsets of } T$.

A $k$-subset of $T$ can be constructed in one of 2 mutually exclusive ways:

- Include $x$: choose $(k-1)$ elements from $S$, throw in $x$.
  
  # Ways = \binom{n}{k-1} 

- Do not include $x$: choose $k$ elements from $S$.
  
  # Ways = \binom{n}{k}$
i, by sum rule

\[ \# \text{k-subsets of } T = \binom{n}{k-1} + \binom{n}{k} \]

.: LHS = RHS, i.e.

\[ \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \]

///.

\[ \lim_{k \to 0} \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

Proof: LHS & RHS both count the \# of subsets of an n-set.

///.
The Binomial Theorem

Observe

\[(x+y)^0 = 1\]

\[(x+y)^1 = 1 \cdot x + 1 \cdot y\]

\[(x+y)^2 = 1 \cdot x^2 + 2 \cdot x \cdot y + 1 \cdot y^2\]

\[(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2 \cdot y + 3 \cdot x \cdot y^2 + 1 \cdot y^3\]

\[(x+y)^4 = 1 \cdot x^4 + 4 \cdot x^3 \cdot y + 6 \cdot x^2 \cdot y^2 + 4 \cdot x \cdot y^3 + 1 \cdot y^4\]

Thus let \(n \in \mathbb{N}\), \(x, y \in \mathbb{R}\). Then

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]
Proof: (combinatorial)

When

\[(x+y)^n = (x+y)(x+y) \ldots (x+y)\]

\[\text{in fully expanded, but before}\]

\[\text{like terms are combined we}\]

\[\text{have } 2^n \text{ terms, each of}\]

\[\text{form } x^{n-k}y^k \text{ for some}\]

\[0 \leq k \leq n.\]

\[\text{Continue}\]