Ex. let \( A_1, A_2, A_3, \ldots \) be an infinite sequence of sets. Then

\[
\forall n \geq 1 : \quad \bigcup_{k=1}^{n} A_k = \bigcap_{k=1}^{n} \overline{A_k}
\]

i.e. \( A_1 \cup A_2 \cup \ldots \cup A_n = A_1 \cap A_2 \cap \ldots \cap A_n \)

**Proof:**

1. \( n = 1 \). \( \overline{A_1} = \overline{A_1} \), it is true.

   (note: \( n = 2 \) is De Morgan's law)

2. \( n = 2 \). \( \overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2} \)
\[ \forall n \geq x \colon P(n) \implies P(n+1) \]

Let \( n \geq x \) be arbitrary. Assume \( P(n) \) is true:

\[ \bigcup_{k=1}^{n} A_{k} = \bigcap_{k=1}^{n} \overline{A}_{k} \]

We must show \( P(n+1) \) is true:

\[ \bigcup_{k=1}^{n+1} A_{k} = \bigcap_{k=1}^{n+1} \overline{A}_{k} \]

So

\[ \bigcup_{k=1}^{n+1} A_{k} = \left( \bigcup_{k=1}^{n} A_{k} \right) \cup A_{n+1} \]

\[ = \left( \bigcup_{k=1}^{n} A_{k} \right) \cap A_{n+1} \]

\[ = \left( \bigcap_{k=1}^{n} \overline{A}_{k} \right) \cap \overline{A}_{n+1} \]

\[ \text{by D.N.K. law} \]

\[ \text{by the ind. hyp.} \]
Exercises:

0. \( \forall n \geq 1 : \bigcap_{k=1}^{n} \overline{A_k} = \bigcup_{k=1}^{n} \overline{A_k} \)

Let \( p_1, p_2, p_3, \ldots \) be an infinite sequence of \( p \)-points.

0. \( \forall n \geq 1 : \bigcap_{k=1}^{n} \left( \bigwedge_{k=1}^{n} p_k \right) = \bigvee_{k=1}^{n} \left( \neg p_k \right) \)

0. \( \forall n \geq 1 : \bigcap_{k=1}^{n} \left( \bigvee_{k=1}^{n} p_k \right) = \bigwedge_{k=1}^{n} \left( \neg p_k \right) \)

Notation: \( \bigvee_{k=1}^{n} p_k = p_1 \lor p_2 \lor p_3 \lor \ldots \lor p_n \)

\( \bigwedge_{k=1}^{n} p_k = p_1 \land p_2 \land \ldots \land p_n \)
5.2 Strong induction \& well ordering

Recall:

\textbf{Theorem (PMI)}

Given any \( P : \mathbb{Z}^+ \rightarrow \{ \text{false, true} \} \),

\[
( P(1) \land \forall n (P(n) \Rightarrow P(n+1)) ) \Rightarrow \forall n \ P(n)
\]

is true.

\textbf{Proof is Based on:}

\underline{Well ordering property of \( \mathbb{Z}^+ \):}

Any non-empty subset of \( \mathbb{Z}^+ \) contains a least element.
Note: This is false if we replace \( \mathbb{Z}^+ \) by \( \mathbb{Z} \) or by \( \mathbb{Z}^- \), also not true if we replace "least" by "greatest".

Proof of P.M.I.:

Assume both \( P(1) \) and \( \forall n \,(P(n) \rightarrow P(n+1)) \) are true. We must show that \( \forall n \, P(n) \) is also true.

Let \( S = \{ n \in \mathbb{Z}^+ \mid \neg P(n) \} \), i.e. \( S \) is the set of positive integers for which \( P(n) \) is false. It is sufficient to show \( S = \emptyset \).
Assume, to get a contradiction that \( S \neq \emptyset \). By the well ordering property, \( S \) contains a least element, call it \( m \). Thus \( meS \) but \( m-1 \notin S \).

Since \( P(1) \) is true \( 1 \in S \).
\[ \therefore m \neq 1 \quad ; \quad m > 1 \quad ; \quad m \geq 2 \quad ; \quad m-1 \geq 1 \]
\[ \therefore m-1 \in \mathbb{Z}^+ \]

Since \( \forall n (P(n) \rightarrow P(n+1)) \) is true we have \( P(m-1) \rightarrow P(m) \) is true, by \( \text{viv. instantation with } n = m-1 \).
Since \( m-1 \notin S \), we have \( \neg P(m-1) \) is true.

\( \therefore P(m) \) is true, by modus ponens.

\( \therefore m \notin S \).

we have the \( \forall \) that \( m \notin S \) and \( m \notin S \).

Therefore our assumption was false, thus \( S = \emptyset \) and hence \( \forall n \neg P(n) \) is true.
Variations on induction

The inductive step can be re-parametrized.

I. Show $P(n_0)$

II. Show $\forall n > n_0 : P(n-1) \Rightarrow P(n)$

contrast with

IIa. Show $\forall n \geq n_0 : P(n) \Rightarrow P(n+1)$

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

II. $P(1) : 1 = \frac{1(1+1)}{2}, \text{i.e. } 1 = 1 \vee$
II b. \( \forall n \geq 1: \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \uparrow \)

Ind. hyp. Ind. conclusion

Let \( n \geq 1 \) be chosen arbitrarily,

Assume \( \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \)

must show \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \)

So

\[
\sum_{k=1}^{n} k = \left( \sum_{k=1}^{n-1} k \right) + n
\]

\[
= \frac{n(n-1)}{2} + n \uparrow \text{by the Ind. hyp.}
\]

\[
= \frac{n(n-1) + 2n}{2}
\]

\[
= \frac{n(n+1)}{2} \cdot \text{IIIB}
\]
Exercise: re-prove all previous examples using II b.

Another variation:

Strong induction or the 2nd principle of mathematical induction.

II. Prove $P(1)$

II. Prove $\forall n \geq 1 : (P(1) \land P(2) \land \ldots \land P(n-1)) \rightarrow P(n)$

i.e. $\forall n \geq 1 : (\forall k < n : P(k)) \rightarrow P(n)$

ind. hyp.
In this case the induction hypothesis is the stronger assumption: \( P(1) \land P(2) \land \ldots \land P(n-1) \) are all true.

**Theorem**

Every positive integer can be written as a product of (zero or more) primes.

*Note:* this is existence part of FTA. The other half is uniqueness.
Proof:

Let \( P(n) = \text{"n is a product of primes"} \)

i. \( P(1) \) is true since 1 is an empty product of primes.

ii. \( \forall n \geq 1 : P(1) \land P(2) \land \ldots \land P(n-1) \Rightarrow P(n) \)

Let \( n \geq 1 \) be arbitrary. Assume for all \( k \) in the range \( 1 \leq k < n \) that \( P(k) \) is true, i.e. \( k \) is a product of primes.
we must show \( P(n) \) is true:
i.e. \( n \) is a product of primes.

If \( n \) is itself prime, then
\( n \) is a product with \( 1 \) only and so
\( P(n) \) is true.

If \( n \) is composite, then
\[
  n = a \cdot b
\]
where \( 1 < a < n \) and \( 1 < b < n \).

By the inductive hypothesis both \( P(a) \) and \( P(b) \) are true, i.e. both
\( a \) and \( b \) are products of primes.
Therefore \( n = 2 \cdot 3 \) is also a product of primes, i.e. \( \mathcal{P}(n) \) is true.

The result holds for all \( n \geq 1 \) by the 2nd PMI.

Recall: weak induc.

\[ \mathcal{P}(1) \land \forall n \geq 1 \left( \mathcal{P}(n-1) \implies \mathcal{P}(n) \right) \]

\[ \mathcal{P}(1) \implies \mathcal{P}(2) \]

\[ \therefore \mathcal{P}(2) \implies \mathcal{P}(3) \]

\[ \therefore \mathcal{P}(3) \]

\[ \vdots \]
strong ind. also in like an infinite

\begin{align*}
P(1) & \land \forall n \geq 1 (P(1) \land \ldots \land P(n-1)) \implies P(n) \\
P(1) & \implies P(2) \quad \quad n = 2 \\
P(2) & \\
P(1) & \land P(2) \\
P(1) & \land P(2) \implies P(3) \quad (n = 3) \\
P(3) & \\
P(1) & \land P(2) \land P(3) \\
P(1) & \land P(2) \land P(3) \implies P(4) \quad (n = 4) \\
P(4) & \\
\end{align*}
another variation: multiple base cases

\[ \forall n \geq 2 : \exists x \geq 0 \exists y \geq 0 : n = 2x + 3y \]

I. Two base cases

- \( P(2) : 2 = 2 \cdot 1 + 3 \cdot 0 \)
- \( P(3) : 3 = 2 \cdot 0 + 3 \cdot 1 \)

II. Show \( \forall n > 3 : P(2) \land P(3) \land \ldots \land P(n-1) \Rightarrow P(n) \)

\[
\begin{align*}
&\text{largest smallest} \\
&\text{base base} \\
&\text{case case}
\end{align*}
\]
Let $n > 3$, assume for $2 \leq k < n$ that:

$$\exists x_k \geq 0 \exists y_k \geq 0 : k = 2x_k + 3y_k$$

we must show:

$$\exists x \geq 0 \exists y \geq 0 : n = 2x + 3y$$

now $n > 3 \Rightarrow 2 \leq n - 2 < n$, so by the inductive hyp. there exist $x' \geq 0, y' \geq 0$ such that

$$n - 2 = 2x' + 3y'$$

Let $x = x' + 1, y = y'$. Then $x \geq 0, y \geq 0$ and

$$n = (n - 2) + 2 = (2x' + 3y') + 2 = (2x' + 2) + 3y' = 2(x' + 1) + 3y' = 2x + 3y.$$