Section 2.2: Set Operations

The union of sets $A$, $B$:

$$A \cup B = \{ x \in U \mid x \in A \lor x \in B \}$$

- **Universe**: set of all objects under discussion

**Venn Diagram**:

```
  A  B
```

$U$
The intersection of sets $A, B$:

$A \cap B = \{x \in U \mid x \in A \land x \in B\}$

We say $A$ and $B$ are disjoint if $A \cap B = \emptyset$. 

\[ U \]

\[ A \]

\[ B \]
The set difference \( A - B \):

\[
A - B = \{ x \in U \mid x \in A \land x \notin B \}
\]

\[
\uparrow
\]

\[7(x \in B)\]

\[
B - A = \{ x \in U \mid x \in B \land x \notin A \}
\]

\[
\uparrow
\]

\[7(x \in A)\]
Other notation: $A \setminus B$ for $A - B$

**Theorem**

If $A$ and $B$ are finite, so are $A \cup B$, $A \cap B$, $A - B$, $B - A$.

Also

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[ |A - B| = |A| - |A \cap B| \]

\[ |B - A| = |B| - |A \cap B| \]
(*) is a special case of the 
Inclusion-Exclusion Principle (PIE)

The complement of $A$:

$$
\overline{A} = \mathcal{U} - A = \{x \in \mathcal{U} \mid x \notin A\}
$$

Note: $\overline{A}$ is meaningless unless $\mathcal{U}$ has been specified.
The symmetric difference of $A$, $B$: 

$A \oplus B = \{ x \in U | x \in A \text{ xor } x \in B \}$

$A \oplus B = (A \cup B) - (A \cap B)$

$= (A - B) \cup (B - A)$
Note: Two sets are equal iff they have exactly the same members: \( A \subseteq B \) and \( B \subseteq A \).

Set Identities (Table 1, p. 130)

**Associative laws:**

\[ A \cup (B \cup C) = (A \cup B) \cup C \]
\[ A \cap (B \cap C) = (A \cap B) \cap C \]

**Distributive laws:**

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
DeMorgan's laws:

\[
\overline{A \cup B} = \overline{A} \cap \overline{B} \\
\overline{A \cap B} = \overline{A} \cup \overline{B}
\]

Double complement:

\[
\overline{\overline{A}} = A
\]

Proof of 2nd DeMorgan:

\[
\overline{A \cup B} = \{ x \mid x \notin (A \cup B) \} \\
= \{ x \mid x \in \overline{A} \cap \overline{B} \}
\]
\[
\{ x \mid \neg (x \in A \lor x \in B) \}
\]

\[
= \{ x \mid \neg (x \in A) \land \neg (x \in B) \}
\]

\[
= \{ x \mid x \notin A \land x \notin B \}
\]

\[
= \overline{A} \cap \overline{B}
\]

Exercise: Prove all identities on p. 130 in this manner.
We can also use membership tables.

**Proof 2nd Distributive:**

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

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<th>B\cap C</th>
<th>A\cup(B\cap C)</th>
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Exercise: Use membership table to determine whether symmetric difference is associative.

\[ A \oplus (B \oplus C) = (A \oplus B) \oplus C \]

Picture for 2nd Dist.
Representations of sets

We often wish to represent sets in a computer program.
One way: Start with a fixed collection of objects, indexed 1 to n.

\[ U = \{ x_1, x_2, x_3, \ldots, x_n \} \]

Then represent subsets of U by bitstrings of length n.
S \subseteq U is defined by

- \( x_i \in S \) iff \( i^{th} \) bit is 1
- \( x_i \notin S \) iff \( i^{th} \) bit is 0
\[ \mathcal{U} = \{ 1, 2, 3, 4 \} \]

0000 \rightarrow \emptyset

1111 \rightarrow \mathcal{U}

1011 \rightarrow \{ 1, 3, 4 \}

0101 \rightarrow \{ 2, 4 \}

1000 \rightarrow \{ 1, 2 \}

bitwise or \rightarrow union

bitwise and \rightarrow intersection

bitwise xor \rightarrow \text{symm. diff}

bitwise negation \rightarrow \text{complement}
2.3 Functions

Definition

A function (also map, mapping, or transformation) consists of three things:

1. A set $A$ called Domain.
2. A set $B$ called Codomain.
3. A rule $f$ that assigns to each $x \in A$ a unique $y \in B$.

Notation: if $f$ maps $x$ to $y$ we write $y = f(x)$

Notation: $f: A \rightarrow B$
Uniqueness:

not unique
not a function

or
does not violate uniqueness
is a function.