Ex. $P_0$ dollars are deposited in a bank at interest rate $r$ (compounded annually). Determine the amount after $n$ years: $P_n$.

\[ P_1 = P_0 + rP_0 = (1+r)P_0 \]
\[ P_2 = P_1 + rP_1 = (1+r)P_1 = (1+r)^2 P_0 \]
\[ P_3 = P_2 + rP_2 = (1+r)P_2 = (1+r)^3 P_0 \]

\[ P_n = (1+r)^n P_0 \]

Solves
\[ P_n = P_{n-1} + rP_{n-1} \]
Find a recurrence for

\[1, 7, 25, 79, 241, 727, 2185, 6559, \ldots\]

\[
\begin{array}{cccccccc}
6 & 18 & 54 & 162 & 486 & 1458 & 4374 \\
3 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}
\]

\[a_n = a_{n-1} + 3(a_{n-1} - a_{n-2})\]

\[
\begin{cases}
  a_n = 4a_{n-1} - 3a_{n-2} & (n \geq 2) \\
  a_0 = 1 \\
  a_1 = 7
\end{cases}
\]

Also check that

\[\begin{array}{c}
\boxed{a_n = 3^{n+1} - 2}
\end{array}\]

is the solution.

\[\text{RHS} = 4a_{n-1} - 3a_{n-2} = 4\left(3^{n-2} - 2\right) - 3\left(3^{n-1} - 2\right) = 3^n - 3^{n-1} + 6 = 4 \cdot 3^n - 3^n - 2 = 3^n - 2 = a_n = \text{LHS}\]
Define a summation (or series) in the sum of a sequence (finite or infinite).

Notation:

\[ \sum_{i=0}^{n} a_i = a_0 + a_1 + \ldots + a_n \]

Upper limit

\[ \sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n \]

Lower limit

\[ i, i \text{ are indices of summation} \]
\[ \sum_{k=1}^{5} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \]

\[ \sum_{i=0}^{6} 2^i = 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \]

\[ \sum_{i=0}^{6} 2^i = 2 + 4 + 8 + 16 + 32 + 64 + 128 \]

**Notation:** \[ \sum_{k \in S} a_k \]

**Example:** \[ S = \{2, 4, 6\} \]

\[ \sum_{k \in \{2, 4, 6\}} k^2 = 2^2 + 4^2 + 6^2 \]
Some special formulas:

1. \[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]
2. \[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]
3. \[ \sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 \]
4. \[ \sum_{k=0}^{n} a^k = \begin{cases} a(n+1) & \text{if } n=1 \\ a \left( \frac{r^{n+1} - 1}{r-1} \right) & \text{if } r \neq 1 \end{cases} \]
Proof of (1):

Let \( S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \)

\[ S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \]

\[ 2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \]

\[ \therefore 2S = n(n+1) \]

\[ \therefore S = \frac{n(n+1)}{2} \]

Proof of (4):

Let \( S = \sum_{k=0}^{n} ar^k \) where \( r \neq 1 \).

\[ S = a(1 + r + r^2 + \cdots + r^{n-1} + r^n) \]

\[ rS = a(r + r^2 + r^3 + \cdots + r^n + r^{n+1}) \]
\[ \therefore rS - S = a \left( r^{n+1} - 1 \right) \]

\[ \therefore (r-1)S = a \left( r^{n+1} - 1 \right) \]

\[ \therefore S = a \left( \frac{r^{n+1} - 1}{r-1} \right). \]

\[ \sum_{k=0}^{20} 3 \cdot 2^k = 3 \left( \frac{2^{21} - 1}{2 - 1} \right) \]

\[ = 3 \left( 2^{21} - 1 \right) = 6291453 \]

\[ \sum_{k=1}^{1000} k = \frac{1000 \cdot 1001}{2} = 500 \cdot 1001 = 500500 \]

\[ \sum_{k=1}^{10} k^2 = \frac{10(10+1)(2\cdot10+1)}{6} = 385 \]
\[
\sum_{k=1}^{20} 3 \cdot 2^k = -3 + 3 + \sum_{k=1}^{20} 3 \cdot 2^k
\]

\[
= -3 + \left( \sum_{k=0}^{20} 3 \cdot 2^k \right)
\]

\[
= -3 + 3 \left( 2^{21} - 1 \right)
\]

\[
= -3 + 629145 - 3
\]

\[
= 629145 - 6
\]

\[
= 629145 - 0
\]
2.5 Cardinality of Sets

**Defn**

Two sets $A, B$ (not necessarily finite) are said to have the same cardinality if there exists a bijection $f: A \rightarrow B$

notation: $|A| = |B|$

This extends notion of sets having 'same size' to infinite sets.
\[ \exists x \quad |N| = |\mathbb{Z}^+| \]

\[ N = \{0, 1, 2, 3, \ldots\} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \]

Define \( f : N \rightarrow \mathbb{Z}^+ \) by \( f(n) = n + 1 \).

Note \( f^{-1} : \mathbb{Z}^+ \rightarrow N \) is \( f^{-1}(m) = m - 1 \).

\[ \therefore f \text{ is a bijection} \]

\[ \exists x \quad |N| = \text{\{even numbers\}} \]

\[ N = \{0, 1, 2, 3, \ldots\} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \]
\[ f(n) = 2n \]
\[ f^{-1}(m) = \frac{m}{2} \]

\[ \text{even numbers} \quad \therefore f \text{ is bijective} \]
Exercise \[ |\mathbb{N}| = |\text{odd numbers}| \]

\[ \text{Ex} \quad |\mathbb{N}| = |\mathbb{Z}| \]

<table>
<thead>
<tr>
<th>\text{even}</th>
<th>\text{odd}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

\[ f(n) = \begin{cases} \frac{n}{2} & \text{even} \\ \frac{n + 1}{2} & \text{odd} \end{cases} \]
Remark

If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$. i.e. show

\[ A \xrightarrow{b_i} B \xrightarrow{b_i} C \]

\[ g \circ f \]

i.e. show composition of bijections is a bijection.

Show: \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\)

\[
(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ i_B \circ g^{-1} = g \circ g^{-1} = \text{id}_C
\]
Define a set $S$ is called countable if either

- $S$ is finite: $|S| < \infty$
- $|S| = |\mathbb{N}|$ (or $= |\mathbb{Z}^+| = |\mathbb{Z}| = \ldots$)

Note in 2nd case we sometimes say $S$ is countably infinite.

If $S$ is not countable, it's called uncountable.
countable sets:
\[ \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}^+, \mathbb{Z}, \ldots \]

Informally: a set is countable if its elements can be arranged in a seq., i.e. an infinite list.

\[ \mathbb{Z}^+ \times \mathbb{Z}^+ \] is countable.
Exercise: Write down the formula for the above map.

\[ f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+ \]

Exercise
show that any subset of a countable set is countable.

hint: it's sufficient to show any subset of \( \mathbb{Z}^+ \) is countable.

Exercise
show that \( \mathbb{Q}^+ \) is countable.

hint: show there is a bijection from \( \mathbb{Q}^+ \) to a subset of \( \mathbb{Z}^+ \times \mathbb{Z}^+ \)
Theorem:

\( \mathbb{R} \) is uncountable.