2.3 functions

A function (map, mapping, transformation) consists of 3 things:

1.) A set A called Domain.
2.) A set B called Codomain.
3.) A rule \( f \) that assigns to each element \( x \in A \) a unique element \( y \in B \).
Notation: \( y = f(x) \)

- \( y \) is called the image of \( x \) under \( f \).
- \( x \) is called a preimage of \( y \) under \( f \).

Notation: \( f : A \rightarrow B \)

**Uniqueness:**

Not a fcn.

OK
Define the range of \( f \) as the set of all images under \( f \):

\[
\text{Range}(f) = \{ y \in B \mid \exists x \in A : y = f(x) \} = \{ f(x) \mid x \in A \}
\]

\[\text{Ex}: f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = x^2\]

\[
\text{Range}(f) = \{0, 1, 4, 9, 16, 25, \ldots \}
\]
\[ f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 \]

\[ \text{Range}(f) = \{ y \in \mathbb{R} \mid y \geq 0 \} = [0, \infty) \]

**Defn**

Let \( f : A \to B \), let \( S \subseteq A \). The **image of \( S \) under \( f \)** is

\[ f(S) = \{ f(x) \mid x \in S \} \subseteq \text{Range}(f) \]

**Defn**

Let \( f : A \to B \), \( T \subseteq B \). The **preimage of \( T \) under \( f \)**

\[ f^{-1}(T) = \{ x \in A \mid f(x) \in T \} \]
\[ f^{-1}(x) = \begin{cases} 
 2 & \text{if } x \leq 2 \\
 6 & \text{if } x > 2 
\end{cases} \]

\[ S = \{ (x, y) : 2 \leq x \leq 3) \}
\]

\[ \text{Range}(f) = \{(x, y) : 1 \leq x \leq 2, 4 \leq y \leq 6\} \]
A function $f: A \rightarrow B$ is called **one-to-one (injective)** if

$$\forall x_1, x_2 \in A : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

equivalently:

$$\forall x_1, x_2 \in A : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

![Diagram of injective function](image1)

**injective**

![Diagram of not injective function](image2)

**not injective**
Ex. \( f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 \)
not injective.

\[ \exists x_1, x_2 \in \mathbb{R} : x_1 \neq x_2 \land f(x_1) = f(x_2) \]

Let \( x_1 = 1, \quad x_2 = -1 \). Then

\[ f(x_1) = 1^2 = 1 = (-1)^2 = f(x_2) \].

Ex. \( g: \mathbb{R} \to \mathbb{R}, \quad g(x) = 2x + 3 \)
is injective

Proof:

Pick \( x_1, x_2 \in \mathbb{R} \) arbitrarily. Assume \( g(x_1) = g(x_2) \).

i.e. \( 2x_1 + 3 = 2x_2 + 3 \).
\[ \therefore 2x_1 = 2x_2 \]
\[ \therefore x_1 = x_2 \].
**Definitions**

We say that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is **strictly increasing** if \( x < y \Rightarrow f(x) < f(y) \), and **strictly decreasing** if \( x < y \Rightarrow f(x) > f(y) \). It is called **strictly monotone** if it is either strictly inc. or dec.

**Theorem**

If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is strictly monotone, then \( f \) is injective.
**Defn**

$f : A \to B$ is called **onto (surjective)** if $\text{Range}(f) = B$, i.e.

$$\forall y \in B, \exists x \in A : y = f(x)$$

\[ \begin{array}{c c}
A & B \\
\text{Surjective} & \text{not Surjective} \\
\end{array} \]

**Ex.** $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$  **not Surjective**

$\exists y \in B, \forall x \in A : y \neq f(x)$.

**Pl: let** $y = -1$. Then $\forall x, f(x) = x^2 \geq 0$, so $f(x) \neq -1$. 
Ex\[ g : \mathbb{R} \to \mathbb{R}, \quad g(x) = 2x + 3 \]
is surjective.

Pl. must show \( \forall y \in \mathbb{R}, \exists x \in \mathbb{R}: y = 2x + 3 \).

Pick any \( y \in \mathbb{R} \) arbitrarily. Let
\[
x = \frac{y - 3}{2}.
\]
Then
\[
g(x) = 2 \left( \frac{y - 3}{2} \right) + 3 = (y - 3) + 3 = y.
\]

Defn. \( f : A \to B \) is called a \underline{one-to-one correspondence} (bijective) if it is both injective \& surjective.

Ex \( g(x) = 2x + 3 \) is bijective
\[
g : \mathbb{R} \to \mathbb{R}
\]
Define the composition of functions \( f: A \to B \) and \( g: B \to C \) as the function \( g \circ f: A \to C \) given by

\[
g \circ f (x) = g( f(x) ) \quad \forall x \in A
\]

Diagram:

\[\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{g \circ f} & & & & \\
x & \mapsto f(x) & \mapsto g(f(x)) & \mapsto \end{array}\]
Let $A = B = C = \mathbb{R}$

$f(x) = x^2$, $g(x) = 2x + 3$

$g \circ f(x) = g(f(x)) = g(x^2) = 2x^2 + 3$

$f \circ g(x) = f(g(x)) = f(2x + 3) = (2x + 3)^2$

In general, $f \circ g \neq g \circ f$, even when both compositions are defined.

**Exercise:**

Show that if $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$

then $h \circ (g \circ f) = (h \circ g) \circ f$.
Invertibility

Let A, B be sets. Define

\[ i_A : A \rightarrow A \text{ by } i_A(x) = x \quad (\forall x \in A) \]

\[ i_B : B \rightarrow B \text{ by } i_B(y) = y \quad (\forall y \in A) \]

identity functions on A, B respectively.

Observe that if \( f : A \rightarrow B \) then

\[ f \circ i_A = f \text{ and } i_B \circ f = f \]

Define

\( f : A \rightarrow B \) is invertible iff there exists \( g : B \rightarrow A \) such that

\[ f \circ g = i_B \text{ and } g \circ f = i_A \]

\( g \) is called an inverse of \( f \).
i.e.

\[ f(g(y)) = y \quad \text{and} \quad g(f(x)) = x \]

for all \( x \in A, \ y \in B \).

Therefore, if \( f \) is invertible, then its inverse \( g \) is unique.

**Proof:**

Suppose there exist two functions \( g_1 : B \to A \) and \( g_2 : B \to A \) satisfying

\[ f \circ g_1 = i_B \quad \text{and} \quad g_1 \circ f = i_A \]

and

\[ f \circ g_2 = i_B \quad \text{and} \quad g_2 \circ f = i_A \]
Then

\[ g_1 = i_A \circ g_1 \]

\[ = (g_2 \circ f) \circ g_1 \]

\[ = g_2 \circ (f \circ g_1) \]

\[ = g_2 \circ i_B \]

\[ = g_2 \]

\[ \therefore g_1 = g_2 \]

We now speak of the inverse of \( f \), write \( f^{-1} = g \)

\[ f^{-1} : B \to A \]