6.1 Recurrence Relations

Recall that a \( k \)th order recurrence relation (or just recurrence) is an equation which gives the \( n \)th term of a sequence as a function of the \( k \) preceding terms:

\[
x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-k})
\]

If \( k \) initial terms \( x_0, x_1, \ldots, x_{k-1} \) are specified, then such an expression uniquely determines the sequence \( (x_n)_{n=0}^{\infty} \).

A sequence \( (a_n)_{n=0}^{\infty} \) which when substituted into the left and right sides of (1) results in an identity for all \( n \geq 0 \) is said to solve or satisfy the recurrence (1).

i.e. \( (a_n) \) is a solution to (1) if

\[
\forall n \geq 0 : a_n = f(a_{n-1}, \ldots, a_{n-k})
\]

In general, a recurrence such as (1) has infinitely many solutions.
Ex. \( x_n = 5x_{n-1} - 6x_{n-2} \)

We show that \( a_n = 2^n \) solves this recurrence.

\[
\begin{align*}
RHS &= 5a_{n-1} - 6a_{n-2} \\
&= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} \\
&= (5 \cdot 2 - 6) \cdot 2^{n-2} \\
&= 4 \cdot 2^{n-2} \\
&= 2^n \\
&= a_n \\
&= LHS
\end{align*}
\]

**Exercise:** Prove that \( b_n = 3^n \) also solves this recurrence, and in fact so does any sequence of the form \( (\alpha \cdot 2^n + \beta \cdot 3^n) \) for any constants \( \alpha \) and \( \beta \).

Ex. The Fibonacci recurrence \( F_n = F_{n-1} + F_{n-2} \).

The unique sequence satisfying this recurrence and the initial conditions \( F_0 = 0, \ F_1 = 1 \) is:

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]

**Exercise:** Prove the above statement.

(Hint: Use the fact that \( \frac{1 + \sqrt{5}}{2} \) are roots of the quadratic \( r^2 - r - 1 = 0 \).)
The quantity \( \frac{1 + \sqrt{5}}{2} \) is often called the Golden Ratio.

Recurrences may be used to model problems in economics and population growth.

Example: \$5000 is placed in an account earning 6\% APR compounded monthly. What amount will be in the account after 10 years?

Define
\[
A_n = \text{Amount (\$) after } n \text{ months}
\]

Then
\[
A_0 = 5000
\]

And
\[
A_n = A_{n-1} + \left( \frac{0.06}{12} \right) A_{n-1} = \left( 1.005 \right) A_{n-1}
\]

Thus
\[
A_1 = \left( 1.005 \right) \cdot 5000
\]
\[
A_2 = \left( 1.005 \right) \cdot \left( 1.005 \right) \cdot 5000 = \left( 1.005 \right)^2 \cdot 5000
\]
\[
A_3 = \left( 1.005 \right) \cdot \left( 1.005 \right) \cdot \left( 1.005 \right) \cdot 5000 = \left( 1.005 \right)^3 \cdot 5000
\]
\[
\vdots
\]
\[
A_n = \left( 1.005 \right)^n \cdot 5000
\]

Exercise: Prove that \( \left( 1.005 \right)^n 5000 \) solves the recurrence \( A_n = \left( 1.005 \right) \cdot A_{n-1} \) and satisfies \( A_0 = 5000 \).
After 10 years (120 months) we have

\[ A_{120} = (1.005)^{120} \cdot 5000 = \$9046.98 \]

Recurrences can also be used to model counting problems.

Ex. Let \( T_n \) denote the number of bit strings of length \( n \) which contain 2 consecutive zeros. Find a recurrence for the sequence \( (T_n)_{n=0}^\infty \).

First observe: \( T_0 = 0, T_1 = 0, T_2 = 1, T_3 = 3, \ldots \)

We can construct a bit string of length \( n \) which contains 2 consecutive zeros by performing exactly one of the following substages.

1) Form a bit string of length \( n-1 \) which contains 2 consecutive zeros, then append a 1.

\[ \underbrace{xx \ldots x}_{n-1} 1 : T_{n-1} \text{ ways} \]
(2) Form a bit string of length \( n-2 \) which contains \( z \) consecutive zeros, then append 10.

\[
\begin{array}{c}
\underbrace{XX\ldots X}_{n-2} 10 : R_{n-2} \text{ ways}
\end{array}
\]

(3) Form an arbitrary bit string of length \( n-2 \), then append 00.

\[
\begin{array}{c}
\underbrace{XX\ldots X}_{n-2} 00 : 2^{n-2} \text{ ways}
\end{array}
\]

Note that every bit string of length \( n \) containing 00 falls into exactly one of the above classes. By the sum rule:

\[
R_n = R_{n-1} + R_{n-2} + 2^{n-2}
\]

Using initial terms \( R_0 = R_1 = 0 \) we obtain:

\[
\begin{align*}
R_2 &= 0 + 0 + 2^0 = 1 \\
R_3 &= 1 + 0 + 2^1 = 3 \\
R_4 &= 3 + 1 + 2^2 = 8 \\
R_5 &= 8 + 3 + 2^3 = 19 \\
R_6 &= 19 + 8 + 2^4 = 43 \\
\end{align*}
\]
EXERCISE

Show that

\[ R_n = \left( \frac{3-\sqrt{5}}{2} \right) \left( \frac{1-\sqrt{5}}{2} \right)^n - \left( \frac{3+\sqrt{5}}{2} \right) \left( \frac{1+\sqrt{5}}{2} \right)^n + 2^n \]

is the unique solution to \( R_n = R_{n-1} + R_{n-2} + 2^{n-1} \) satisfying \( R_0 = R_1 = 0 \).

HINT: As for the Fibonacci sequence, use the fact that \( \frac{1 \pm \sqrt{5}}{2} \) are roots of the quadratic \( x^2 - x - 1 = 0 \).
6.2 Solving Linear Recurrence Relations

A $k^{th}$ order linear homogeneous recurrence relation with constant coefficients is an equation of the form

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \ldots + c_k x_{n-k}$$

where $c_1, \ldots, c_k \in \mathbb{R}$ and $c_k \neq 0$.

- **Linear**: RHS is a 1st degree polynomial in $x_{n-1}, \ldots, x_{n-k}$
- **Homogeneous**: Each term on RHS is of degree 1
- **Constant Coefficients**: $c_1, \ldots, c_k$ are not functions of $n$.

Examples

<table>
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<tr>
<th></th>
<th>Order</th>
<th>Linear</th>
<th>Homogeneous</th>
<th>Const. Coeff.</th>
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<tbody>
<tr>
<td>$F_n = F_{n-1} + F_{n-2}$</td>
<td>2</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
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<tr>
<td>$x_n = 2x_{n-1}$</td>
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<td>yes</td>
<td>yes</td>
<td>yes</td>
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<tr>
<td>$B_n = B_{n-1} + B_{n-2} + 2^{n-2}$</td>
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<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>$Y_n = Y_{n-1} + Y_{n-2}^2$</td>
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<td>NA</td>
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<tr>
<td>$a_n = n \cdot a_{n-1} + n$</td>
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<td>yes</td>
<td>no</td>
<td>no</td>
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