4.2 The Pigeonhole Principle

Ex.
A drawer contains a dozen brown socks and a dozen black socks (unmatched.) A man takes out socks at random in the dark. How many socks must be removed to be sure he has (at least) 2 of the same color?
Answer: 3

What is the smallest number of socks whose removal would guarantee that the man has 6 of the same color?
Answer: 11

Theorem: Pigeonhole Principle (PP)
If k+1 (or more) objects are placed in k boxes, then (at least) one box contains (at least) 2 objects.

Proof
Assume, to get a contradiction, that all k boxes contain fewer than 2 objects.
"All boxes contain at most 1 object"
"The number of objects is at most the number of boxes k. This contradicts that there are k+1 objects."
Ex.
In any group of 13 people, at least 2 were born in the same month.

Ex.
How many students must be enrolled in a university to guarantee that at least 100 were born in the same month?
Answer: 1189

Theorem: Generalized Pigeonhole Principle (GPP)
If n (or more) objects are placed in k boxes, then (at least) one box contains (at least) \( \lceil \frac{n}{k} \rceil \) objects.

In the previous example, we seek the smallest n for which

\[ \lceil \frac{n}{12} \rceil = 100 \]

i.e.

99 < \( \frac{n}{12} \) \leq 100

i.e.

12.99 < n \leq 12.100

The smallest such n is obviously

\[ n = 12.99 + 1 = 1189 \]
Proof: (of GPP)

Assume, to get a contradiction, that each box contains at most \( \left\lceil \frac{n}{k} \right\rceil - 1 \) objects. Then the number of objects \( N \) must satisfy:

\[
N \leq k \cdot \left( \left\lceil \frac{n}{k} \right\rceil - 1 \right) = k \cdot \frac{n}{k} - n = n - n = 0,
\]

which says \( N < N \), a contradiction.

so some box must contain at least \( \left\lceil \frac{n}{k} \right\rceil \) objects.

Note: we used \( \left\lceil x \right\rceil < x + 1 \) \( \forall x \in \mathbb{R} \).

Ex.

There are 38 different time periods during which classes at a university may be scheduled. If there are 677 different classes, how many rooms are needed?

By the GPP, at least one period must have \( \left\lceil \frac{677}{58} \right\rceil = \left\lceil 11.8 \right\rceil = 12 \) classes assigned to it. Therefore, at least 12 rooms are necessary.
Ex:
Find the smallest \( n \) such that in any group of \( n \) people, at least 100 have the same birthday.

We seek the smallest \( n \) such that
\[
\frac{n}{366} = 100
\]

i.e.
\[
99 < \frac{n}{366} = 100
\]
\[
366 \cdot 99 < n \leq 366 \cdot 100
\]
\[
\therefore n = 366 \cdot 99 + 1 = 36,235
\]

Theorem
In any group of 6 people in which each pair are either friends or enemies, there must exist either a group of 3 mutual friends or a group of 3 mutual enemies.

It is helpful to state this theorem in terms of graph theory.

A graph \( G \) consists of a pair \((V, E)\) of sets called vertices and edges, respectively. We require \( V \) to be a finite set and each edge \( e \in E \) is an unordered pair of vertices:
\[
e = \{x, y\} \subseteq V.
\]
We write \( e = xy \).
Ex.\[ \begin{array}{c}
1 \\
\times \\
3 \\
\times \\
2 \\
\end{array} \quad \sqrt[4]{1, 2, 3, 4} \]
\[E = \{12, 13, 14, 23\}.\]

We say that vertices 1 and 2 are adjacent, that edge 12 joins 1 to 2, and that edge 12 is incident with vertex 1.

The complete graph on n vertices has an edge joining each pair of vertices.

\[
\begin{align*}
&n = 3 \\
&n = 4 \\
&n = 5 \\
&n = 6
\end{align*}
\]

The preceding theorem can now be stated as follows: let G be the complete graph on 6 vertices, and suppose each edge of G is colored either blue (friendship) or red (enmity). Then G must contain either a blue triangle or a red triangle.
Proof of Theorem.
Let A be a person in the group. The remaining 5 people fall into one of 2 classes: $F = \{\text{friends of } A\}$, $E = \{\text{enemies of } A\}$.

By the P.P. at least $\left\lceil \frac{5}{2} \right\rceil = 3$ people are in the same class.

Case 1: $|F| \geq 3$
Say B, C, D are friends of A. If any two of these, say B, C, are friends (of each other) then A, B, C forms a group of 3 mutual friends. Otherwise B, C, D form a group of 3 mutual enemies.

Case 2: $|E| \geq 3$
Follows from Case 1 by swapping 'friend' $\leftrightarrow$ 'enemy'.

In both cases there is either a group of 3 mutual friends or 3 mutual enemies. //
**NOTE:** 6 is the smallest number of people for which this conclusion is true. For example, it is possible that in a group of 5 people (in which each pair are either friends or enemies), there exists neither 3 mutual friends nor 3 mutual enemies.

**DEFN:** Let \( G = (V,E) \) and \( x \in V \). The degree of \( x \) is the number of edges incident with \( x \).

**Ex**

\[
\begin{align*}
\text{deg}(1) &= 2, & \text{deg}(2) &= 4, \\
\text{deg}(4) &= 3, & \text{deg}(6) &= 0
\end{align*}
\]

**Theorem**

Let \( G \) be a graph on \( n \geq 2 \) vertices. Then \( G \) contains 2 vertices of like degree.

**Proof:** Exercise

Consider subgraph \( (V-V_0, E) \) where \( V_0 = \{ \text{vertices of degree 0} \} \), then apply the pigeonhole principle.