THEOREM

If $S$ is finite then so is $P(S)$ and

$$|P(S)| = 2^{1|S|}$$

(Hence the name 'Power' set.)

Ex. $S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Thus $|P(S)| = 8 = 2^3 = 2^{1|S|}$

Recall a set is an unordered collection.
An ordered collection of objects is called an ordered $n$-tuple

**Notation:** $(x_1, x_2, x_3, \ldots, x_n)$

If $n = 2$ we speak of an ordered pair.

Note: $(x_1, x_2) = (y_1, y_2)$ iff $x_1 = y_1$ and $x_2 = y_2$.

For instance: $(u, v) = (1, 2)$ iff $u = 1$, $v = 2$. 
Define

**The Cartesian product of two sets** \( A, B \) is the set of all ordered pairs \( (x, y) \) where \( x \in A \) and \( y \in B \).

**Nomination:** \( A \times B \)

\[
A \times B = \{ (x, y) \mid x \in A \land y \in B \}
\]

*Example:* \( A = \{1, 2, 3\} \), \( B = \{a, b\} \)

\[
A \times B = \{ (1, a), (1, b), (2, a), (2, b), (3, a), (3, b) \}
\]

**Theorem:**
If \( A \) and \( B \) are finite, so is \( A \times B \), and

\[
|A \times B| = |A| \cdot |B|
\]

More generally, **the Cartesian product** of \( n \) sets \( A_1, A_2, \ldots, A_n \) is

\[
A_1 \times A_2 \times \cdots \times A_n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in A_i \text{ for } 1 \leq i \leq n \}
\]

And

\[
|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|
\]
Russell's Paradox (Prob 30 p. 86)

Let \( S = \{ \text{all sets} \} \). Observe \( S \in S \).

But \( \emptyset \notin \emptyset \), so some sets have themselves as members, and some do not.

Define: \( R = \{ A \in S \mid A \notin A \} \)

Is \( R \in R \) or \( R \notin R \)?

Observe that \( R \in R \Rightarrow R \notin R \) and \( R \notin R \Rightarrow R \in R \). Thus

\[ R \in R \iff R \notin R \]

We avoid this paradox by simply not allowing the construction of \( S \), or any other set which is a member of itself. This ad-hoc approach is often called Naive Set Theory.
1.7 Set Operations

**Defn:**
The union of sets $A, B$ is the set
\[ A \cup B = \{ x \in U | x \in A \vee x \in B \} \]

**Venn Diagram:**

**Defn:**
The intersection of sets $A, B$ is the set
\[ A \cap B = \{ x \in U | x \in A \cap x \in B \} \]

**Ex.**
\[ A = \{1, 2, 3\} \quad B = \{2, 3, 4, 5\} \]
\[ A \cup B = \{1, 2, 3, 4, 5\} \]
\[ A \cap B = \{2, 3\} \]
We call $A$ and $B$ disjoint if $A \cap B = \emptyset$

If $A$ and $B$ are finite, then so are $A \cup B$ and $A \cap B$; and

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This is a special case of the **Principle of Inclusion-Exclusion (PIE)**, which is an important counting technique.

Define the **set difference** $A - B$ in the set

$$A - B = \{ x \in U \mid x \in A \land x \notin B \}$$
DEFINITION:
The complement of $A$ is the set
\[
\overline{A} = U - A = \{ x \in U \mid x \notin A \}
\]

Example: $U = \{1, \ldots, 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

$A - B = \{1, 2\}$, $\overline{A} = \{5, \ldots, 10\}$

Note: $\overline{A}$ is meaningless unless $U$ has been specified.

Read set identifier Table 1, p. 89

$A \cup (B \cup C) = (A \cup B) \cup C$ \quad \text{associative laws}

$A \cap (B \cap C) = (A \cap B) \cap C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ \quad \text{distributive laws}

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{De Morgan's laws}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Proof of 1st DeMorgan:

\[ \overline{A \cup B} = \{ x \mid x \notin A \cup B \} \]
\[ = \{ x \mid \neg x \in A \cup B \} \]
\[ = \{ x \mid \neg (x \in A \lor x \in B) \} \]
\[ = \{ x \mid \neg (x \in A \land \neg x \in B) \} \]
\[ = \{ x \mid x \in \overline{A} \land x \in \overline{B} \} \]
\[ = \overline{A} \land \overline{B} \]

Identities can also be proved by membership tables, e.g. 2nd Distributive

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<th>B \cap C</th>
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Thus \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

\[ 
\begin{array}{c}
\text{Notation:} \\
\text{Given sets } A_1, A_2, \ldots, A_n \\
A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i \\
A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i.
\end{array}
\]

**DeMorgan**

The **symmetric difference** of \( A, B \) is

\[ A \oplus B = \{ x \in U \mid x \in A \oplus x \in B \} \]

**Example**

Show \( A \oplus B = (A \cup B) - (A \cap B) \).
(1.8) Functions

**Definition**

A function consists of three things:
1. A set A called the **domain**
2. A set B called the **codomain**
3. A rule \( f \) which assigns to each element of A, a **unique** element of B.

If \( x \in A \) is assigned to \( y \in B \) we write \( y = f(x) \). \( y \) is called the **image** of \( x \) under \( f \) and \( x \) is called the **preimage** of \( y \) under \( f \).

**Notation:** \( f : A \rightarrow B \)

**Uniqueness:**

**Not a function**

**OK**
The range of $f$ is the set of all images under $f$.

$$\text{Range}(f) = \{ y \in B \mid \exists x \in A : y = f(x) \}$$

$$= \{ f(x) \mid x \in A \}$$

**Ex.** $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = x^2$

$$\text{Range}(f) = \{ 0, 1, 4, 9, 16, 25, \ldots \}$$

**Ex.** $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$

$$\text{Range}(f) = [0, \infty) = \{ y \in \mathbb{R} \mid y \geq 0 \}$$

**Defn.** Let $f : A \to B$, $S \subseteq A$. The image of $S$ under $f$ is $f(S) = \{ f(x) \mid x \in S \}$. 

**Note:** $f(S) \subseteq f(A) = \text{Range}(f) \subseteq B$. 