4.3 Permutations and Combinations

**Definition:** A permutation of a set is simply an ordered arrangement of its elements.

Ex. \( S = \{1, 2, 3\} \)

Permutations of \( S \):
\[
123, 132, 213, 231, 312, 321
\]

**Definition:** A \( k \)-permutation of a set \( S \) is an ordered arrangement of \( k \) elements of \( S \) \((0 \leq k \leq |S|)\).

Ex. \( S = \{1, 2, 3\} \)

\( k \)-permutations of \( S \), \( k = 0, 1, 2, 3 \):
\[
\begin{align*}
k = 0 & : \emptyset \quad P(3,0) = 1 \\
k = 1 & : 1, 2, 3 \quad P(3,1) = 3 \\
k = 2 & : 12, 21, 13, 31, 23, 32 \quad P(3,2) = 6 \\
k = 3 & : 123, 132, 213, 231, 312, 321 \quad P(3,3) = 6
\end{align*}
\]

Notation: \( P(n, k) = \# \text{ \( k \)-permutations of an \( n \) element set} \).

**Theorem**
\[
P(n, k) = \frac{n!}{(n-k)!}
\]
For \( 0 \leq k \leq n \).
This follows directly from the Product Rule. Note that a k-permutation of S is just a string of length k from the alphabet S, where no symbol is repeated.

Also note if |S| = n, then an n-permutation of S is just a permutation of S, and \( P(n, n) = n! \).

**Definition:** A k-combination of S is an unordered collection of k elements of S, i.e., a k element subset of S. \( 0 \leq k \leq |S| \).

**Ex.** \( S = \{1, 2, 3\} \)

k-combinations of \( S \), \( k = 0, 1, 2, 3 \):

- k = 0: \( \emptyset \) \( C(3, 0) = 1 \)
- k = 1: \( \{1\}, \{2\}, \{3\} \) \( C(3, 1) = 3 \)
- k = 2: \( \{1, 2\}, \{1, 3\}, \{2, 3\} \) \( C(3, 2) = 3 \)
- k = 3: \( \{1, 2, 3\} \) \( C(3, 3) = 1 \)

**Notation:** \( C(n, k) = \# \text{k-combinations of an n element set} \).

**Other Notation:** \( (n, k) \) Read 'n-choose-k'

Also called binomial coefficients.
**Theorem**

\[ C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

for \( 0 \leq k \leq n \).

**Proof:**

Let \( |S| = n \). A \( k \)-Permutation of \( S \) can be constructed by performing the following subtasks:

- Choose a \( k \)-Element subset of \( S \) in any of \( C(n,k) \) ways.
- Choose a permutation of those \( k \) elements in any of \( k! \) ways.

By the Product Rule,

\[ P(n,k) = C(n,k) \cdot k! \]

\[ \therefore \frac{n!}{(n-k)!} = C(n,k) \cdot k! \]

\[ \therefore C(n,k) = \frac{n!}{k!(n-k)!} \]
Ex. How many bit strings of length 10 contain exactly 6 1's?

\[
\begin{array}{cccccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10}
\end{array}
\]

In how many ways can we choose which 6 of the 10 positions will be occupied by 1's? In how many ways can we choose a 6 element subset of the set \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)?

\[C(10, 6) = \frac{10!}{6! \cdot 4!} = 210\]

Note we could just as well choose where to place the 4 0's:

\[C(10, 4) = \frac{10!}{4! \cdot 6!} = 210\]

**Theorem**

\[\binom{n}{k} = \binom{n}{n-k}\]

**Proof:**

\[\binom{n}{n-k} = \frac{n!}{(n-k)! \cdot (n-(n-k))!} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}\]
There are many identities involving \( C(n,k) \) and \( P(n,k) \). They generally have two kinds of proofs:

- **Algebraic**: Manipulate known identities.
- **Combinatorial**: Argue that LHS and RHS count the very same thing.

Here is a combinatorial proof of the preceding theorem.

**Proof**

Let \( |S| = n \), and define for \( 0 \leq k \leq n \):

\[ S_k = \{ k \text{ element subsets of } S \} \subseteq \mathcal{P}(S). \]

Define \( f : S_k \to S_{n-k} \) by \( f(A) = A = S - A \) for \( A \subseteq S_k \). Observe that \( f(f(A)) = f(S - A) = f(A) = A \). Thus, \( f \) is invertible and \( f^{-1} = f \). Thus, \( f \) is a bijection.

\[ |S_k| = |S_{n-k}|. \]

\[ \binom{n}{k} = \binom{n}{n-k}, \]