\[ x_{k+1} = x_k - 4\Delta x_k + \Delta u_k \]

1) (5 points) Given the difference equation above, find the equilibrium point(s) as a function of \( u_k \)

\[ x_{k+1} = x_k - 4\Delta x_k + \Delta u_k \quad \text{for equilibrium point } x_{k+1} = x_k \]

\[ 4\Delta x_k = \Delta u_k \]

\[ x_k = \frac{u_k}{4} \]

2) (5 points) Determine the stability of the equilibrium point(s) that you found in question (1). Hint: \( \frac{d(x_k - 4\Delta x_k + \Delta u_k)}{dx} = 1 - 4\Delta \) \( x_{k+1} = f(x_k) = x_k - 4\Delta x_k + \Delta u_k \)

\[ \frac{df}{dx} = \frac{d(x_k - 4\Delta x_k + \Delta u_k)}{dx_k} = 1 - 4\Delta \quad \text{for stability } \quad |1 - 4\Delta| < 1 . \]

\[ \text{and is } \quad 0 < \Delta < \frac{1}{4} \quad \text{for stability} \]

3) (5 points) You are given an initial starting point of \( x_0 = 2.5 \), a time step of \( \Delta = 0.2 \), and a constant input of \( u_k = 8 \). Calculate the first 5 points the trajectory:

\[ [x_1, x_2, x_3, x_4, x_5] \]

From #2 above, this should be stable and we have \( x_k = \frac{8}{4} = 2 \).

\[ x_{k+1} = x_k - 4\Delta x_k + \Delta u_k \quad u_k = 8 \]

\[ \Delta = 0.2 \]

\[ x_{k+1} = (1 - 4(0.2)) x_k + (0.2)(8) \]

\[ \text{for } x_0 = 2.5 \]

\[ x_{k+1} = 0.2 \times x_k + 1.6 \]

From #2 above, this should be stable and we have \( x_k = \frac{8}{4} = 2 \).

\[ x_1 = (0.2)(2.5) + 1.6 = 2.1 \]

\[ x_2 = (0.2)(2.1) + 1.6 = 2.02 \]

\[ x_3 = (0.2)(2.02) + 1.6 = 2.004 \]

\[ x_4 = (0.2)(2.004) + 1.6 = 2.0008 \]

\[ x_5 = (0.2)(2.0008) + 1.6 = 2.00016 \]
4) (5 points) Complete the MATLAB function that computes and plots the trajectory of this equation:

```
function [x]=simMidterm(xo,delta,Tend,u)

% xo – initial condition
% delta – time step
% Tend – final time
% u – constant input
%
x(1) = xo;
t(1) = 0;
for k = 2:1, % count in terms of Δ steps
    t(k) = t(k-1) + delta;
    x(k) = x(k-1) - 4*delta*x(k-1) + delta*u;
end
plot(t,x);
```

For questions (5)-(9), you decide to add feedback to drive this system to a desired position $x_{des}$ by using the control law: $u_k = -G(x_k - x_{des})$

5) (5 points) Write the new difference equation $x_{k+1} = f(x_k, x_{des})$

For new equation:

$x_{k+1} = x_k - 4\Delta x_k + \Delta a_k$

$u_k = -G(x_k - x_{des})$

$x_{k+1} = x_k - 4\Delta x_k - \Delta G(x_k - x_{des})$

$x_{k+1} = x_k - 4\Delta x_k - \Delta G x_k + \Delta G x_{des}$

$X_{k+1} = (1-4\Delta - \Delta G) x_k + \Delta G x_{des}$

6) (5 points) Find the equilibrium point(s) for the new system.

For equilibrium:

$$x_{k+1} = x_k = x_k$$

$$x_k = (1-4\Delta - \Delta G) x_k + \Delta G x_{des}$$

$$x_k - 4\Delta x_k + \Delta G x_{des} = \Delta G x_{des}$$

$$4\Delta x_k = \Delta G x_{des}$$

$$x_k = \frac{\Delta G}{4\Delta} x_{des}$$
7) (5 points) For a sample time of $\Delta = 0.25$, find the limits of $G$ for stability. Hint:
\[
\frac{d(Ax)}{dx} = A \text{ for any constant } A \\
\chi_{k+1} = F(\chi_k) = (1-4\Delta - 6\Delta C)\chi_k + 6\Delta C x_{des}
\]
\[
\frac{dG}{dx} \frac{d((1-4\Delta - 6\Delta C)\chi_k + 6\Delta C x_{des})}{dx} = 1 - 4\Delta - 6\Delta C
\]
For stability: $|\frac{dG}{dx}| < 1$, $\Delta = 0.25$ so $\frac{dG}{dx} (1-4(0.25)-0.25G) = -\frac{G}{4}$.
So for stability: $\boxed{-4 < G < 4}$

8) (5 points) Calculate the trajectory for an initial starting point of $x_0 = 0$, a time step $x_{des} = 2$, of $\Delta = 0.25$, and a feedback gain $G = 2$. Calculate the first 5 points the trajectory:
\[
[x_1 \ x_2 \ x_3 \ x_4 \ x_5] \\
\chi_{k+1} = (1-4\Delta - 6\Delta C)\chi_k + 6\Delta C x_{des}
\]
\[
x_{k+1} = 2((1-4(0.25)-2(0.25))\chi_k + (0.25)y(2))
\]
\[
x_{k+1} = \frac{2}{2} \chi_k + 1
\]
$\chi_0 = 0$,
$\chi_1 = \frac{2}{2} \chi_0 + 1 = \frac{3}{2}$,
$\chi_2 = \frac{2}{2} \chi_1 + 1 = \frac{5}{4}$,
$\chi_3 = \frac{2}{2} \chi_2 + 1 = \frac{31}{16}$,
$\chi_4 = \frac{2}{2} \chi_3 + 1 = \frac{19}{8}$,
so $\chi_{k+1} = \frac{19}{8}$.

9) (5 points) Complete the MATLAB function that computes and plots the trajectory of the feedback system (make sure you plot both the trajectory $x$ and the control $u$):

```matlab
function [x]=simMidtermFeedback(xo,delta,Tend,G,xdes)
%
% xo - initial condition
% delta - time step
% Tend - final time
% G - feedback gain
% xdes - desired target
x(1) = xo;
t(1) = 0;
u(1) = G * (x(1) - xdes);
for k = 2:Tend/delta,
    t(k) = t(k-1) + delta;
    u(k) = G * (x(k) - x(1));
    x(k) = x(k-1) - 4*delta*u(k-1) + x(k-1) + u(k-1)*delta;
end
plot(t,x,t,u,','g');
end
```
\[ v_{k+1} = v_k - \frac{\Delta b}{m} v_k - g \Delta \sin \Theta_k + \frac{\Delta F_k}{m} \]

10) **Bonus question** (10 points) You are given the dynamic equations that describe the simplified cruise control of a car above, where \( m \) is the car mass, \( \beta \) is the friction, \( g \) is gravity, \( F \) is the throttle, and \( \Theta_k \) is the angle of the hill. Applying feedback such that \( F_k = -G(v_k - v_{des}) \), you notice that even on flat ground (no hills), the cruise control does not put the car at the desired speed.

a. (5 points) Fix this by adding “FeedForward,” calculate the feed-forward term.

For flat ground (\( \Theta_k = 0 \)), the fixed point is
\[ v_f = \frac{\beta}{m} v_k - g \sin \Theta_k + \frac{G(v_k - v_{des})}{m} \]

This is the form we will need to add back in to zero the steady state error.

\[ F_{feedForward} = b v_{des} \]

b. (5 points) Write the new equation for \( F_k \)

Notice that with the feedback, we have a steady state error.

(when \( \Theta_k = 0 \)).

\[ v_{k+1} = v_k - \frac{\Delta b}{m} v_k - g \Delta \sin \Theta_k + \frac{\Delta F_k}{m} - G(v_k - v_{des}) \]

For \( \Theta_k = 0 \), find steady state.

\[ v_f = \frac{\beta}{m} v_k - \frac{G v_{des}}{m} + \frac{G v_{des}}{m} \]

\[ (\beta + G) v_f = G v_{des} \rightarrow v_f = \left( \frac{G}{\beta + G} \right) v_{des} \]

Now, add the feedforward term back in:

\[ F_k = -G(v_k - v_{des}) + b v_{des} \]

Now a.

\[ v_{k+1} = v_k - \frac{\Delta b}{m} v_k - g \sin \Theta_k + \frac{\Delta F_k}{m} - \frac{G(v_k - v_{des})}{m} + b v_{des} \]