Quiz #2 Dynamic Models — 4:05-4:35PM (calculators allowed)

\[ \dot{x} + 2x - 3x^2 + 5 = u \]

1) (10 points) Given the (non-linear) differential equation above in \( x \), with an input \( u \), turn this into a difference equation of the form \( x_{k+1} = f(x_k, u_k) \).

\[
\Delta \frac{x_{k+1} - x_k}{\Delta} = \frac{x_{k+1} - x_k}{\Delta} = 2x_k - 3x_k^2 + 5 = u_k \]

\[
\quad \frac{x_{k+1} - x_k}{\Delta} = \frac{3x_k^2 - 2x_k - 5 + u_k}{\Delta} \rightarrow \Delta \frac{x_{k+1} - x_k}{\Delta} = x_k + 3\Delta x_k^2 - 2\Delta x_k - 5\Delta + \Delta u_k
\]

\[
\quad \frac{x_{k+1} - x_k}{\Delta} = (1 - 2\Delta) x_k + 3\Delta x_k^2 - 5\Delta + \Delta u_k
\]

2) (10 points) For the difference equation \( x_{k+1} = -2x_k - 2x_k^2 - 4 + u_k \), find the fixed points for a constant \( u_k = 3 \) for all \( k \).

\[
x_k = -2x_k - 2x_k^2 - 1 \rightarrow 0 = -2x_k^2 - 3x_k - 1 \quad \text{multiplier } \lambda = -1.
\]

\[
x_k = \frac{-3 \pm \sqrt{9 + 8}}{4} = \frac{-3 \pm \sqrt{17}}{4} = -3 \pm \frac{1}{4}
\]

3) (6 points) Calculate the trajectory for the difference equation in (2) above for the first six points starting from \( x_0 = -0.25 \), with \( u_k = 3 \) (constant). That is, calculate \( [x_1 \to x_6] \) using your calculator.

\[
x_1 = \frac{-3 - \sqrt{17}}{4} = -0.625
\]

\[
x_2 = \frac{-3 + \sqrt{17}}{4} = -0.5218
\]

\[
x_3 = \frac{-3 - \sqrt{17}}{4} = -0.020
\]

\[
x_4 = \frac{-3 + \sqrt{17}}{4} = -0.0007
\]

\[
x_5 = -0.5
\]

\[
x_6 = -0.5
\]
4) (6 points) Calculate the trajectory for the difference equation in (2) above for the first six points starting from $x_0 = -1.5$, with $u_0 = 3$ (constant). That is, calculate $[x_1 \text{ to } x_6]$ using your calculator.
\[ x_{k+1} = -2x_k - 2x_k^2 - 1 \quad x_0 = -1.5 \]
\[ x_1 = (-2)(-1.5) - 2(-1.5)^2 - 1 = -2.75 \]
\[ x_2 = (-2)(-2.75) - 2(-2.75)^2 - 1 = -8.5 \]
\[ x_3 = (-2)(-8.5) - 2(-8.5)^2 - 1 = -12.85 \]

5) (4 points) Given your results from question (3) and (4) above, comment on the stability and attractiveness of your two fixed points calculated in (2).

- 0.5 is stable and attractive
- 1 is unstable

6) (6 points) Check the stability of the fixed points that you found in part (2) analytically. Hint: \[ \frac{d}{dx}(2x - 2x^2 - 4 + u_k) = -2 - 4x \]

For equation \( x_{k+1} = g(x_k) = -2x - 2x^2 - 4 + u_k \)

Stability: \( \frac{df}{dx} \bigg|_{x = x^*} = -2 - 4(x^*) = -2 - 4(-0.5) = 2 > 1 \) (NOT STABLE \( x^* = -1 \))

\[ \Theta_{k+1} = 2\Theta_k - \Theta_{k-1} - \Delta^2 \left( \frac{g}{l} \right) \sin\Theta - \Delta \left( \frac{b}{ml^2} \right) \left( \Theta_k - \Theta_{k-1} \right) + \Delta^2 \left( \frac{T}{ml^2} \right) \]

7) (5 points) Bonus: Given the equation of motion for a pendulum with an external applied torque, \( T \), find the equilibrium position as a function of \( T \).

\[ \Theta_0 = \sin^{-1} \left( \frac{T}{mg} \right) \]
Quiz #2 Dynamic Models — 4:05-4:35PM (calculators allowed)

UNIVERSITY OF CALIFORNIA, SANTA CRUZ
BOARD OF STUDIES IN COMPUTER ENGINEERING
CMPE-008: ROBOT AUTOMATION
SPRING 2009

NAME: 

\[ \dot{x} + 3x + 2x^2 + 4 = u \]

1) (10 points) Given the (non-linear) differential equation above in \( x \), with an input \( u \), turn this into a difference equation of the form \( x_{k+1} = f(x_k, u_k) \)

\[
\begin{align*}
\dot{x} & \approx \frac{x_{k+1} - x_k}{\Delta} \\
3x + 2x^2 + 4 & \approx \frac{x_{k+1} - x_k}{\Delta} \\
\Rightarrow x_{k+1} - x_k & = -3x_k - 2x_k^2 - 4 + u_k \\
\Rightarrow x_{k+1} & = x_k - 3\Delta x_k - 2\Delta x_k^2 - 4\Delta + \Delta u_k
\end{align*}
\]

2) (10 points) For the difference equation \( x_{k+1} = 0.5x_k + 0.75x_k^2 - 1.25 + 0.25u_k \), find the fixed points for a constant \( u_k = 4 \) for all \( k \).

\[
0.5x + 0.75x^2 - 1.25 + 0.25 = 0
\]

Using quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{1 \pm \sqrt{1^2 - 4(0.5)(-1.25)}}{2(0.5)}
\]

\[
= \frac{1 \pm \sqrt{1 + 2.5}}{1} = \frac{1 \pm \sqrt{3.5}}{1}
\]

\[
= \frac{1 \pm 1.87}{1}
\]

\[
x_1 = -0.87
\]

\[
x_2 = 2.87
\]

3) (6 points) Calculate the trajectory for the difference equation in (2) above for the first six points starting from \( x_0 = -0.5 \), with \( u_k = 4 \) (constant). That is, calculate \([x_1 \text{ to } x_6]\) using your calculator.

\[
x_1 = (0.5)(-0.5) + (0.75)(-0.5)^2 - 1.25 = -0.3125
\]

\[
x_2 = (0.5)(-0.3125) + (0.75)(-0.3125)^2 - 1.25 = -0.3333
\]

\[
x_3 = (0.5)(-0.3333) + (0.75)(-0.3333)^2 - 0.25 = -0.3333
\]

\[
x_4 = -0.3333
\]

\[
x_5 = -0.3333
\]

\[
x_6 = -0.3333
4) (6 points) Calculate the trajectory for the difference equation in (2) above for the first six points starting from \( x_0 = 1.25 \), with \( n = 4 \) (constant). That is, calculate \([x_1 \text{ to } x_6]\) using your calculator.

\[
\begin{align*}
A_3 &= 1.25 \\
B_3 &= 0.5x_2 + 0.75x_1^2 - 0.25 \\
x_6 &= 1.5499 \\
x_5 &= (0.5x_1(1.25) + 0.75x_2(1.25)^2 - 0.25 = 2.3181 \\
x_4 &= 3.125 \\
x_3 &= 2.9391 \\
x_2 &= 1.25 \\
x_1 &= 1.25 \\
x_0 &= 1.25
\end{align*}
\]

5) (4 points) Given your results from question (3) and (4) above, comment on the stability and attractiveness of your two fixed points calculated in (2).

\[ x = \frac{1}{3} \] is stable and attractive

\[ x = 1 \] is unstable

6) (6 points) Check the stability of the fixed points that you found in part (2) analytically. Hint: \( \frac{d}{dx}(0.5x + 0.75x^2 - 1.25 + 0.25u) = 0.5 + 1.5x \)

Stability of \( x = \frac{1}{3} \):
\[
\frac{df}{dx} = 0.5 + 1.5x < \frac{1}{2} \Rightarrow \text{stable}
\]

Stability of \( x = 1 \):
\[
\frac{df}{dx} = 3 > 1 \Rightarrow \text{unstable}
\]

\[ \Theta_{k+1} = 2\Theta_k - \Theta_{k-1} - \Delta^2 \left( \frac{g}{l} \right) \sin \Theta_k - \Delta \left( \frac{b}{ml^2} \right) \left( \Theta_k - \Theta_{k-1} \right) + \Delta^2 \left( \frac{T}{ml^2} \right) \]

7) (5 points) Bonus: Given the equation of motion for a pendulum with an external applied torque, \( T \), find the equilibrium position as a function of \( T \).

\[
\delta \Theta = \frac{T}{m} \Rightarrow \sin \Theta = \frac{T}{m \delta g} \Rightarrow \Theta = \sin^{-1} \left( \frac{T}{m \delta g} \right)
\]