Sec. 3.4.1 Linear Models

\[ x_{n+1} = a x_n + b \]

for \( a \neq 0 \), and/or \( b \neq 0 \).

Given \( x_1 \), calculate \((x_1, x_2, x_3, x_4, \ldots)\)

\[ \Rightarrow \text{plot} ([1:10], x) \rightarrow \text{get a straight line} \]

\( x \) has length 10

Set \( b = 0 \). Model: \( x_{n+1} = a x_n \).

Fixed points:

\[ \text{if } a \neq 0 \rightarrow x^* = a x^* - x^* - x^* \]

\[ 0 = (a - 1) x^* \]

Two cases

(i) \( a = 1 \) \( \Rightarrow x^* \) is any number

(ii) \( a \neq 1 \) \( \Rightarrow x^* = 0 \)
What if $a = 0$

$$x_{n+1} = 0 \quad x_n = 0$$

$$\Rightarrow x^* = 0$$

Definitions

1. $x^*$ is stable if, for any $x_1$, the trajectory remains "near" $x^*$.

2. $x^*$ is stable and attractive if, when $x_1$ is near $x^*$, the trajectory remains near $x^*$ and converges $x_n \to x^*$.

3. $x^*$ is unstable if it's not stable. There is an $x_1$ near $x^*$ for which the trajectory diverges from $x^*$. 
Task 3.4.3

\[ X_{n+1} = \frac{1}{2} X_n \]

\( X_\ast = 0 \) is the fixed point.

What is the stability?

Try \( x_1 = 1 \).

\[ \text{Trajectory: } (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}) \]

\[ \lim_{n \to \infty} X_n = X_\ast \text{ for } \text{stable } \frac{1}{2} \text{ attractive} \]

\[ X_n = \frac{1}{2^{n-1}} X_1 \]

\[ \lim_{n \to \infty} \frac{1}{2^{n-1}} \cdot X_1 = 0 \]

So \( X_\ast = 0 \) is stable \( \frac{1}{2} \) attractive.

\[ X_{n+1} = 2 X_n \]

What is the stability of \( X_\ast = 0 \)?

\( x_1 = 1 \) → (1, 2, 4, 8, 16, 32, ...)
$x_*=0$ is unstable.

\[ x_2 = 2x_1 \quad x_3 = 2x_2 = 4x_1 \]
\[ \vdots \quad x_{n+1} = 2^n x_1 \]

\[ \lim_{n \to \infty} 2^{n-1} x_1 = \infty \]

\[ x_{n+1} = ax_n \]

\[ x_2 = ax_1 \quad x_3 = a \cdot a \cdot x_1 = a^2 x_1 \]

\[ x_{n+1} = a^n x_1 \]

\[ \lim_{n \to \infty} a^n x_1 = \begin{cases} 0, & |a| < 1 \\ \infty, & |a| > 1 \end{cases} \]

When $|a| \neq 1$ for the model $x_{n+1} = ax_n$

$x_*=0$ is the only fixed point of its stable and attractive if $|a| < 1$
Stability? \( x = \frac{1}{2} \) \( \quad \frac{1}{2} \rightarrow \frac{1}{2} \) - not attractive

\[ \begin{align*}
\text{Stable} & : x^* = 0 \quad 2x = 0 \rightarrow x = 0 \\
\text{Unstable} & : x^* = -x \quad x + x^* = x - x = 0
\end{align*} \]

\[ \begin{align*}
\text{when } a = -1, \quad X_n+1 = X_n - X_n = 0 \\
\text{and every } X^* \text{ is stable, not attractive.}
\end{align*} \]

When \( a = 1 \), \( X_{n+1} = X_n \) \( \Rightarrow \) \( X^* \) is unstable. 

\( |a| > 1 \)
For a model

\[ x_{n+1} = f(x_n) \]

with f.p. \( x^* \),

the f.p. is:

\[ \begin{cases} 
\text{stable} & \text{if } |\frac{df}{dx}(x^*)| < 1 \\
\text{attractive} & \text{if } |\frac{df}{dx}(x^*)| < 1 \\
\text{unstable} & \text{if } |\frac{df}{dx}(x^*)| > 1
\end{cases} \]

If \( |\frac{df}{dx}(x^*)| = 1 \), more analysis required.

\[ x_{n+1} = ax_n + b \]

\[ f(x) = ax + b \]

\[ \frac{df}{dx} = a \]
Task 3.4.5

\[ x_{n+1} = x_n^2 \] - a nonlinear model

Find fixed points

Substitute: \[ x_* = x_*^2 \]

Solve: \[ -x_* - x_* \]

\[ 0 = x_*^2 - x_* = x_* (x_1) \]

2 fixed points: \[ x_* = 0, 1 \]

Stability of \[ x_* = 0 \]

\[ x_1 = 0.1 \Rightarrow (0.1, 0.01, 1e-4, 1e-8, \ldots) \]

\[ x_2 = x_1^2 \]

\[ \Rightarrow \text{stable and attractive} \]

Stability of \[ x_* = 1 \]

\[ (1.1, 1.21, \ldots) \Rightarrow \text{unstable} \]