1. Determine whether each of the following distributions belongs to the exponential dispersion family of distributions. Similarly for the two-parameter exponential family of distributions. (For each distribution, the support, parameter space and density function, or probability mass function, are provided).

(a) **Double exponential (or Laplace) distribution.** $y \in \mathbb{R}, \theta \in \mathbb{R}, \sigma > 0,$

$$f(y; \theta, \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|y - \theta|}{\sigma}\right)$$

(b) **Uniform distribution.** $\theta - \sigma < y < \theta + \sigma, \theta \in \mathbb{R}, \sigma > 0,$

$$f(y; \theta, \sigma) = \frac{1}{2\sigma} 1_{(\theta-\sigma, \theta+\sigma)}(y)$$

(c) **Logistic distribution.** $y \in \mathbb{R}, \theta \in \mathbb{R}, \sigma > 0,$

$$f(y; \theta, \sigma) = \frac{\exp((y - \theta)/\sigma)}{\sigma \{1 + \exp((y - \theta)/\sigma)\}^2}$$

(d) **Cauchy distribution.** $y \in \mathbb{R}, \theta \in \mathbb{R}, \sigma > 0,$

$$f(y; \theta, \sigma) = \frac{1}{\pi\sigma \{1 + ((y - \theta)/\sigma)^2\}}$$

(e) **Pareto distribution.** $y \geq \alpha, \alpha > 0, \beta > 0,$

$$f(y; \alpha, \beta) = \frac{\beta \alpha^\beta y^{\beta-1} 1_{(\alpha, \infty)}(y)}{(y+1)\Gamma(\beta)}$$

(f) **Beta distribution.** $0 \leq y \leq 1, \alpha > 0, \beta > 0,$

$$f(y; \alpha, \beta) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)},$$

where $B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is the beta function.

(g) **Negative binomial distribution.** $y = 0, 1, 2, \ldots$ (a non-negative integer), $\alpha > 0, 0 < p < 1,$

$$f(y; \alpha, p) = \frac{\Gamma(y + \alpha)}{\Gamma(\alpha)y!} p^\alpha (1-p)^y,$$

where $\Gamma(c) = \int_0^\infty u^{c-1} \exp(-u)du$ is the gamma function.
2. Consider the linear regression setting where the responses $Y_i$, $i = 1, \ldots, n$, are assumed independent with means $\mu_i = E(Y_i) = x_i^T \beta = \sum_{j=1}^{p} x_{ij} \beta_j$ for (known) explanatory variables $x_{ij}$ and (unknown) regression coefficients $\beta = (\beta_1, \ldots, \beta_p)^T$.

(i) Show that if the response distribution is normal, i.e.,

$$Y_i \overset{ind.}{\sim} f(y_i; \mu_i, \sigma) = \left(2\pi \sigma^2\right)^{-1/2} \exp \left(-\frac{(y_i - \mu_i)^2}{2\sigma^2}\right), \quad i = 1, \ldots, n,$$

then the maximum likelihood estimate of $\beta$ is obtained by minimizing the $L_2$-norm

$$S_2(\beta) = \sum_{i=1}^{n} (y_i - x_i^T \beta)^2.$$

(ii) Show that if the response distribution is double exponential, i.e.,

$$Y_i \overset{ind.}{\sim} f(y_i; \mu_i, \sigma) = (2\sigma)^{-1} \exp \left(-\frac{|y_i - \mu_i|}{\sigma}\right), \quad i = 1, \ldots, n,$$

then the maximum likelihood estimate of $\beta$ is obtained by minimizing the $L_1$-norm

$$S_1(\beta) = \sum_{i=1}^{n} |y_i - x_i^T \beta|.$$

(iii) Show that if the response distribution is uniform over the range $(\mu_i - \sigma, \mu_i + \sigma)$, i.e.,

$$Y_i \overset{ind.}{\sim} f(y_i; \mu_i, \sigma) = (2\sigma)^{-1} \mathbf{1}_{(\mu_i - \sigma, \mu_i + \sigma)}(y_i), \quad i = 1, \ldots, n,$$

then the maximum likelihood estimate of $\beta$ is obtained by minimizing the $L_\infty$-norm

$$S_\infty(\beta) = \max_{i} |y_i - x_i^T \beta|.$$

(iv) Obtain the maximum likelihood estimate of $\sigma$ under each one of the response distributions in (i) - (iii) and show that, in all cases, it is a function of the minimized norm.