(1) Let \( \{X_n\} \) be random variables describing the population size after \( n \) periods of a branching process such that \( X_0 = 1 \) and the distribution of the size of the offspring is \( p(x) \).
   (a) Find an expression relating the variance of \( X_n \) to the moments of \( p(x) \).
   (b) Use this result to show that, if \( m < 1 \), then \( X_n \to 0 \) in probability as \( n \to \infty \).

(2) Let \( \{X_n\} \) be a branching process with \( X_0 = 1 \) and offspring distribution \( \Pr(Z = k) = bc^{k-1} \) for \( k = 1, 2, \ldots \) and \( \Pr(Z = 0) = 1 - b/(1 - c) \).
   (a) Compute the probability of extinction for this branching process.
   (b) Compute the probability generating function for \( X_n \).
   (c) Determine the probability distribution function for \( X_n \).

(3) Compute the probability of extinction for a branching process with a Poisson(2) offspring distribution, if the population at time 0 follows a uniform distribution on the integers between 1 and 10.

(4) The most successful models for automatic gene annotation use hidden Markov models. A (very rough) version assumes that a section of a DNA sequence can be classified either as an intron, an exon or intragenic (both introns and exons are part of the gene). Describe a hidden Markov model for gene annotation and modify the code provided in class to implement it. Use the following information:
   (a) An intragenic region has to be followed by an exon.
   (b) An intron has to be followed by an exon.
   (c) An exon can be followed by either an intron or an intragenic region.
   (d) The average length of both exons and introns is around 100 bp.
   (e) The average length of intragenic regions is 2000 bp.
   (f) Genes have around 5 introns.
   (g) Assuming that bases are ordered as A, G, T and C, the typical proportion of each nucleotide is
      - \( (0.15, 0.15, 0.35, 0.35) \) for introns.
      - \( (0.35, 0.35, 0.15, 0.15) \) for exons.
      - \( (0.25, 0.25, 0.25, 0.25) \) for intragenic regions.

(5) Implement the dynamic programming algorithm to impute the hidden states in the "cheating casino" example discussed in class.

(6) Let \( X_1, \ldots, X_n \) be an i.i.d. sample from a density \( f \). Show that the joint density for \( (X_1, \ldots, X_n) \), the order statistic of the sample, is given by
   \[
   f(x(1), \ldots, x(n)) = n! \prod_{i=1}^{n} f(x(i))
   \]