and conclude that \( d_1 \) is continuous.

\[
i > s \quad (s - 1)d - 1 = |(s)d - (s)d|
\]

Show (a)

\[
(s)d(s) + (s)d - 1 = (s + 1)d = (s)d(s)
\]

Show that (b)

\[
\frac{1}{(s)d - 1}
\]

find (c)

\[
\tilde{(s)}d = (s)d
\]

Prove Lemma 5.4.2.

Prove Lemma 5.4.1.

\section{Problems}

4. **i)** \( X \) is a distribution of the number of robust individuals born in an individual drawn at time \( s \) will be robust. Consider a process starting with a single individual and suppose an individual born in (\( 0, s \)) with probability \( (s)d(s) \). Given that \( \tilde{X} = i \), let \( \tilde{X} = i \).

\[
\tilde{X} = \int_0^s + \psi = (s)X
\]

4. **ii)** Verify the formula

\[
(u + 1)(\frac{1}{u}) \leq X
\]

The number of individuals born in (\( 0, s \)) can be said about the conditional distribution of the birth times of the individuals born in (\( 0, s \)). Given that \( X = i \), what is the distribution of the birth times of the individuals born in (\( 0, s \))?

For a pure birth process with birth parameters \( \lambda \), we can determine the probability of an individual born in (\( 0, s \)).

\[
X + \cdots + X = \sum_{k=1}^{n} X
\]

4. **iii)** Compute the distribution of \( (s)d(s) \) at time \( s \).

4. **iv)** Let \( X \) and \( Y \) denote the number of males and females, respectively, in any interval of length \( t \), which probability of an individual in any interval of length \( t \) is likely to be the same. Suppose further that any individual has a probability of being of the same sex is likely to be the same. Suppose further that any individual has a probability of being of the same sex is likely to be the same.

\[
\begin{align*}
\tilde{X} - \tilde{Y} &= 0 = (\tilde{X})d
\end{align*}
\]

Also,

\[
\begin{align*}
\frac{1}{(s+1)d} \leq X
\end{align*}
\]

Proving the above together gives, for \( i > s \),

\[
\tilde{X} = \int_0^s + \psi = (s)X
\]

We see that \( \tilde{X} = X \) when \( i = n \), this is the same distribution as the original distribution.

\[
A + \cdots + A = \sum_{k=1}^{n} A
\]

Any of the \( A \)'s is the same number which occurs in the same.

Finally, in the case where \( \lambda \) is the same number which occurs in the same.
5.11. For the Yule process:

(a) verify that

\[ P_i(t) = \left( \frac{j-1}{i-1} \right) e^{-(i-1)t} (1 - e^{-i})^{i-1} \]

satisfies the forward and backward equations.

(b) Suppose that \( X(0) = 1 \) and that at time \( T \) the process is replaced by an emigration process in which departures follow a Poisson process of rate \( \mu \). Let \( T \) denote the time taken for the population to vanish. Find the density function of \( T \) and show that

\[ E[T] = e^{(1)/\mu} \]

5.12. Suppose that the “state” of the system can be modeled as a continuous-time Markov chain with transition rates \( \gamma_i = \lambda_i, \tau_i = \mu_i \). When the state of the system is \( i \), “events” occur in accordance with a Poisson process with rate \( \alpha_i, i = 0, 1 \). Let \( N(t) \) denote the number of events in \((0, t)\).

(a) Find \( \lim_{t \to \infty} N(t)/t \).

(b) If the initial state is state 0, find \( E[N(t)] \).

5.13. Consider a birth and death process with birth rates \( \{\lambda_i\} \) and death rates \( \{\mu_i\} \). Starting in state \( i \), find the probability that the first \( k \) events are all births.

5.14. Consider a population of size \( n \), some of whom are infected by a certain virus. Suppose that in an interval of length \( h \), any specified pair of individuals will independently interact with probability \( \lambda h \). If exactly one of the individuals involved in the interaction is infected, the other one becomes infected with probability \( \alpha \). If there is an individual infected at time 0, find the expected time at which the population is infected.

5.15. Consider a population in which each individual independently is born at an exponential rate \( \lambda \) and dies at an exponential rate \( \mu \). In addition, new members enter the population in accordance with a Poisson process with rate \( \theta \). Let \( X(t) \) denote the population size at time \( t \).

(a) What type of process is \( \{X(t), t \geq 0\} \)?

(b) What are its parameters?

(c) Find \( E[X(t)|X(0) = i] \).

5.16. In Example 5.4(D), find the variance of the number of males in the population at time \( t \).
5.22. Find the limiting probabilities for the $M/M/1$ system and determine the condition needed for these to exist.

5.23. If $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are independent time-reversible Markov chains, show that the process $\{X(t), Y(t), t \geq 0\}$ is also.

5.24. Consider two $M/M/1$ queues with respective parameters $\lambda_i, \mu_i, i = 1, 2$. Suppose they both share the same waiting room that has finite capacity $N$. (That is, whenever this room is full, all new arrivals to either queue are lost.) Compute the limiting probabilities there will be $n$ people at the first queue ($1$ being served and the waiting room when $n > 0$) and $m$ at the second. (Hint: result of Problem 5.23.)

5.25. What can you say about the departure process of the stationary $M/M/1$ queue having finite capacity?

5.26. In the stochastic population model of Section 5.6.2:
(a) Show that

$$P(n)q(n, D, D) = P(D, D)q(n, D, n)$$

when $P(n)$ is as given by (5.6.4) with $\alpha_j = (\lambda/\mu)^j \mu^j$.

(b) Let $D(t)$ denote the number of families that die out in $0, 1, \ldots, t$. Assuming that the process is in steady state $D = 0$ at time $t = 0$, what stochastic process is $\{D(t), t \geq 0\}$? What if the population is empty at $t = 0$?

5.27. Complete the proof of the conjecture in the queueing network of Section 5.7.1.

5.28. $N$ customers move about among $r$ servers. The service times for $i$ are exponential at rate $\mu_i$ and when a customer leaves server $i$, it enters the queue (if there are any waiting—or else it enters server $j$, $j \neq i$, with probability $1/(r - 1)$. Let the state be $(n_1, \ldots, n_r)$ with $n_i$ customers at server $i$, $i = 1, \ldots, r$. Show the continuous-time Markov chain is time reversible and find the transition probabilities.

5.29. Consider a time-reversible continuous-time Markov chain having parameters $\mu_i, \nu_j$ and having limiting probabilities $P_j, j \geq 0$. Consider state $0$—say state 0—and consider the new Markov chain, which state 0 an absorbing state. That is, reset $\nu_0$ to equal 0. Suppose time points chosen according to a Poisson process with rate $\Lambda$.

5.30. Prove Theorem 5.7.1.

(a) Prove that a stationary Markov process is reversible if, and only if, it transitions satisfy

$$q(j_1, j_2)q(j_2, j_3) \cdots q(j_{n-1}, j_n)q(j_n, j) = q(j_1, j_n)q(j_n, j_{n-1}) \cdots q(j_2, j_1)q(j_1, j)$$

for any finite sequence of states $i, j_1, j_2, \ldots, j_n$.

(b) Argue that it suffices to verify that the equality in (a) holds for sequences of distinct states.